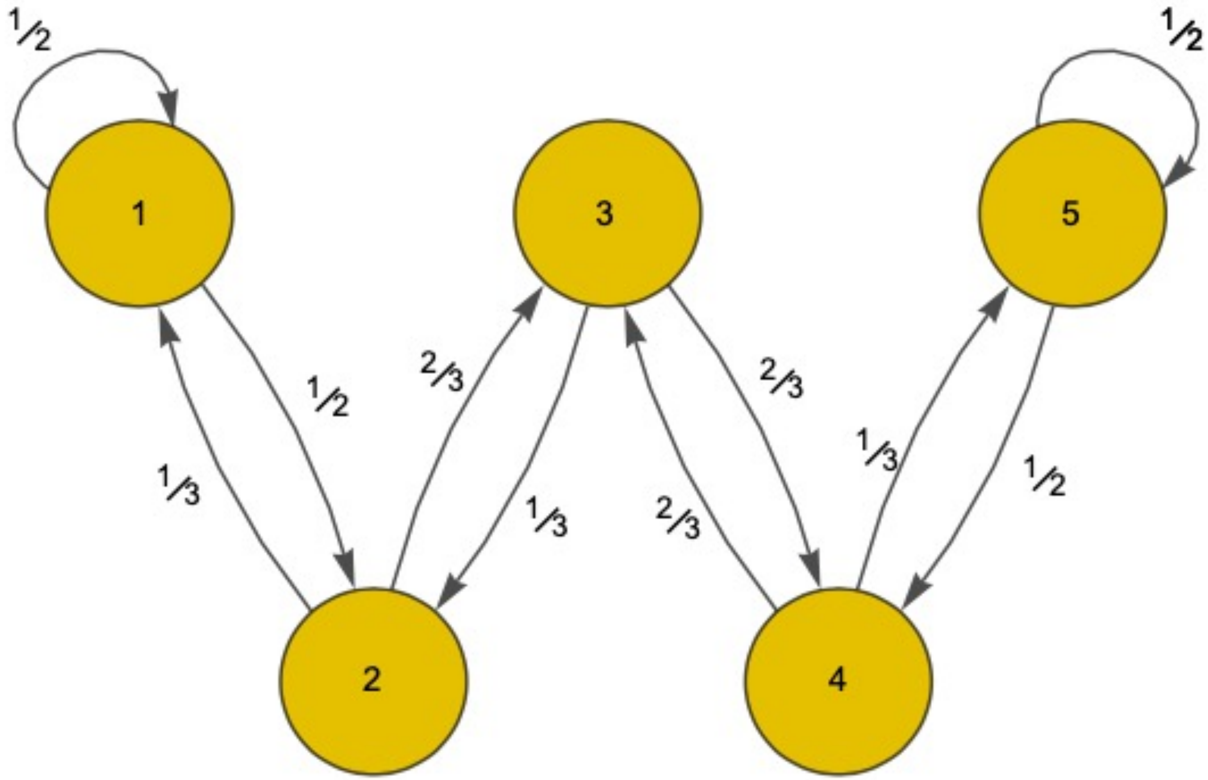
Stochastic Processes: Basic Definitions

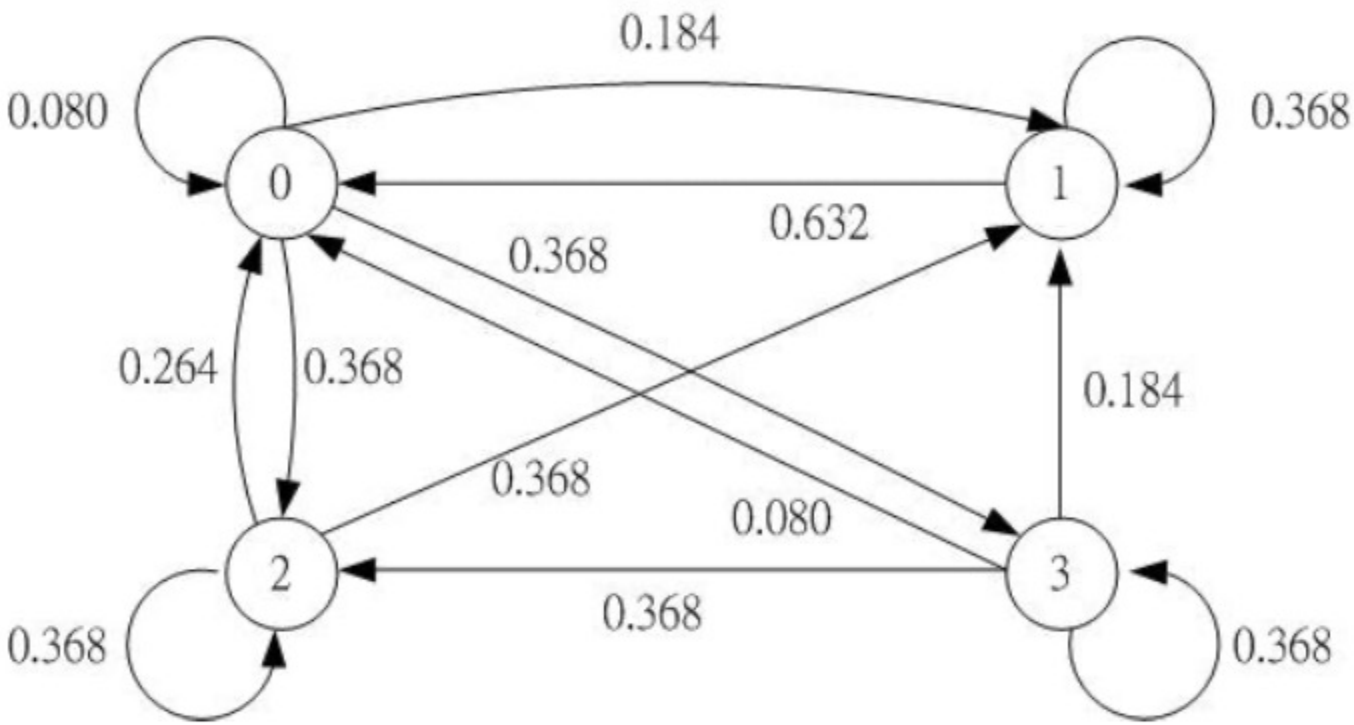
Stochastic process	The value of a variable changes in an uncertain way
Discrete vs. continuous time	When can a variable change? What values can a variable take?
Markov property	Only the current value of a variable is relevant for future predictions No information from past prices or path

A Chain



Markov Chain

- The state transition diagram:



Markov Chain

- Consider time index $n = 0, 1, 2, \ldots$ & time dependent random state X_n
- State X_n takes values on a countable number of states
  - In general denotes states as $i = 0, 1, 2, \ldots$
  - Might change with problem
- Denote the history of the process $\mathbf{X}_n = [X_n, X_{n-1}, \ldots, X_0]^T$
- Denote stochastic process as $\mathbf{X}_{\mathbb{N}}$
- The stochastic process $X_{\mathbb{N}}$ is a Markov chain (MC) if

$$\mathrm{P}\left[X_{n+1} = j \,\middle|\, X_n = i, \mathbf{X}_{n-1}\right] = \mathrm{P}\left[X_{n+1} = j \,\middle|\, X_n = i\right] = P_{ij}$$

- Future depends only on current state X_n

Observations

- Process's history $\mathbf{X}_{n-1}$ irrelevant for future evolution of the process
- Probabilities P_{ij} are constant for all times (time invariant)

- From the definition we have that for arbitrary m

$$\mathrm{P}\left[X_{n+m} \,\middle|\, X_n, \mathbf{X}_{n-1}\right] = \mathrm{P}\left[X_{n+m} \,\middle|\, X_n\right]$$

- X_{n+m} depends only on X_{n+m-1}, which depends only $on X_{n+m-2}$, ... which depends only on X_n

- Since P_{ij}'s are probabilities they're positive and sum up to 1

$$P_{ij} \geq 0 \qquad \sum_{j=1}^{\infty} P_{ij} = 1$$

Matrix Representation

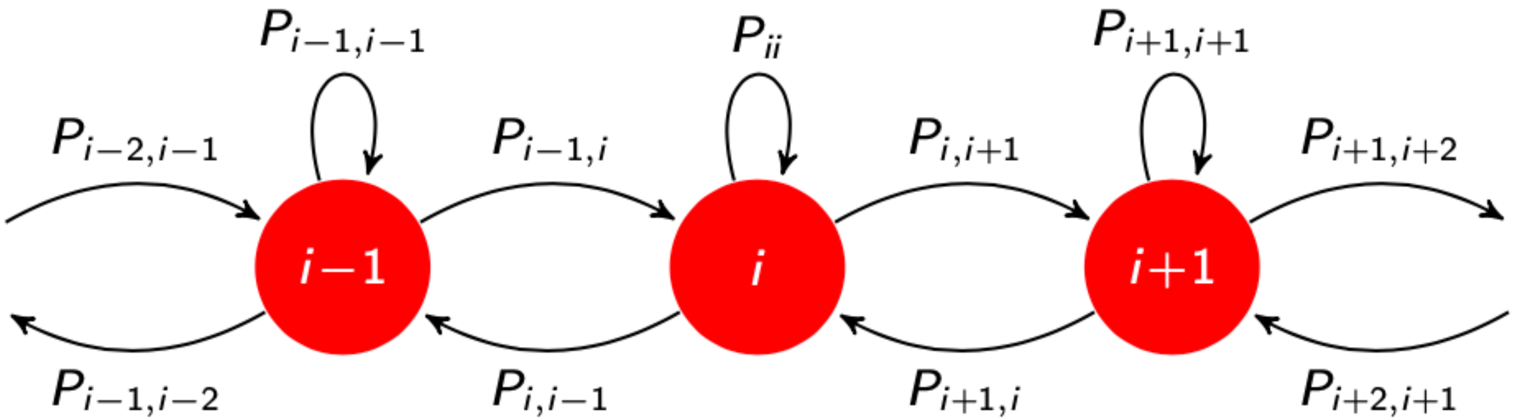
- Group transition probabilities P_{ij} in a "matrix" $\mathbf{P}$

$$\mathbf{P} := \begin{pmatrix} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ P_{i0} & P_{i1} & P_{i2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- Not really a matrix if number of states is infinite

Graph Representation

- A graph representation is also used

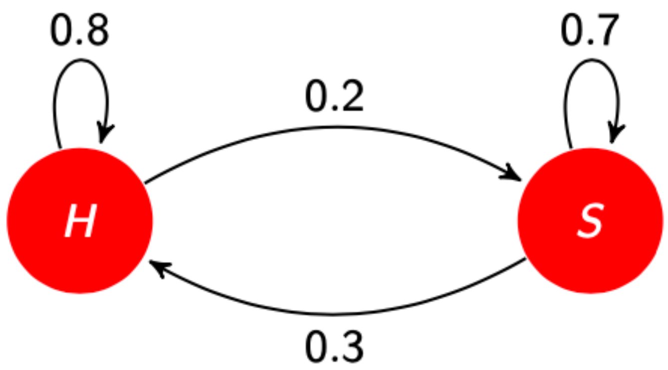


- Useful when number of states is infinite

Happy – Sad

- I can be happy ($X_n = 0$) or sad ($X_n = 1$).
- Happiness tomorrow affected by happiness today only
- Model as Markov chain with transition probabilities

$$\mathbf{P} := \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}$$

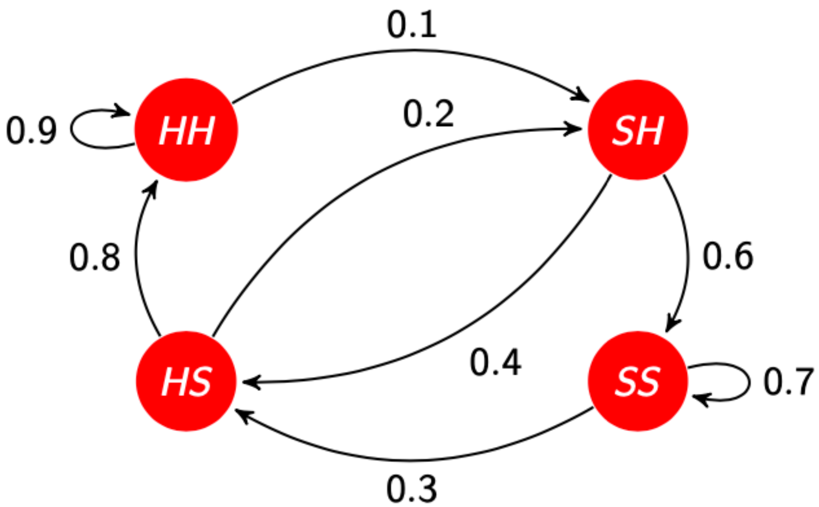


- Inertia $\Rightarrow$ happy or sad today, likely to stay happy or sad tomorrow ($P_{00} = 0.8$, $P_{11} = 0.7$)
- But when sad, a little less likely so ($P_{00} > P_{11}$)

Happy – Sad 2

- Happiness tomorrow affected by today and yesterday
- Define double states HH (happy-happy), HS (happy-sad), SH, SS
- Only some transitions are possible
  - HH and SH can only become HH or HS
  - HS and SS can only become SH or SS

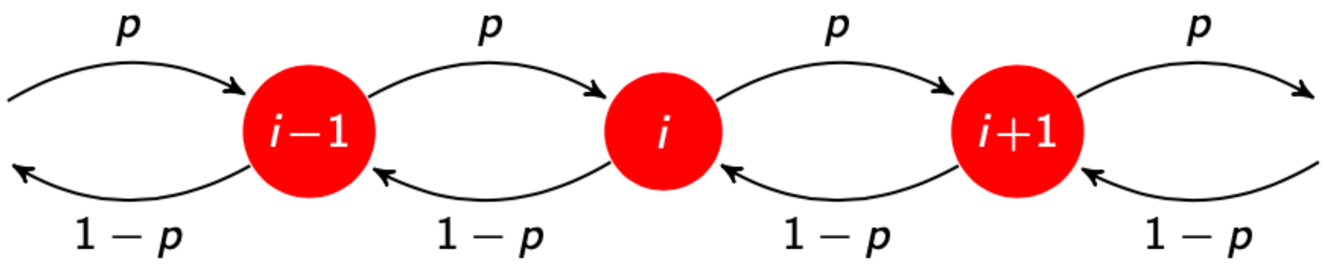
$$\mathbf{P} := \begin{pmatrix} 0.9 & 0.1 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0.8 & 0.2 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 \end{pmatrix}$$



- More time happy or sad increases likelihood of staying happy or sad
- State augmentation $\Rightarrow$ Capture longer time memory

Random Walk

- Step to the right with probability p, to the left with prob. (1-p)



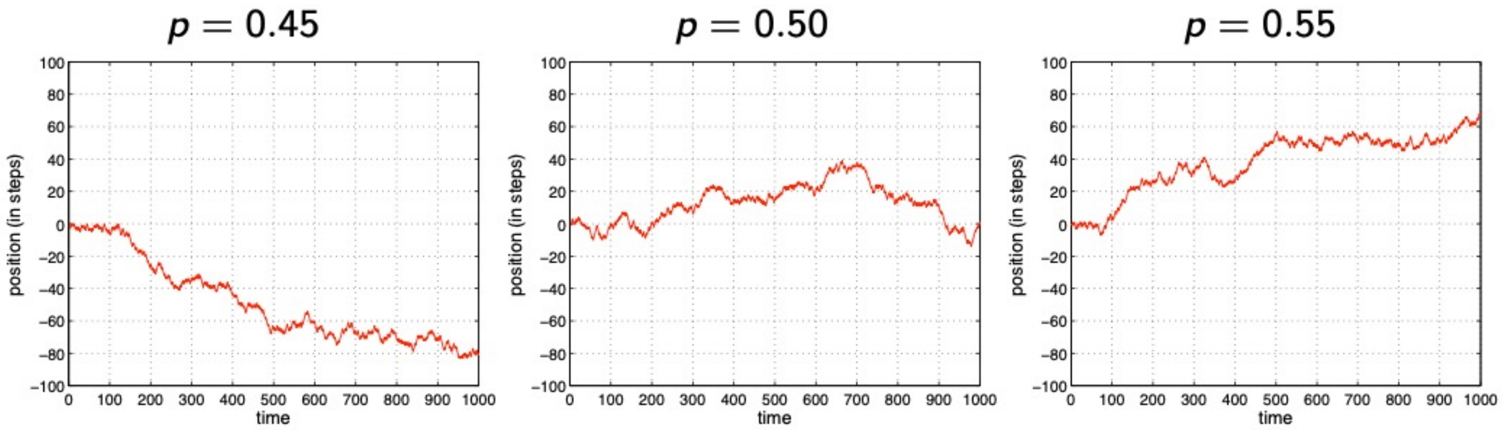
- States are $0, \pm 1, \pm 2, \ldots$, number of states is infinite
- Transition probabilities are

$$P_{i,i+1} = p, \qquad P_{i,i-1} = 1 - p,$$

- $P_{ij} = 0$ for all other transitions

Random Walk Continous

- Random walks behave differently if $p < 1/2$, $p = 1/2$ or $p > 1/2$



- With $p > 1/2$ diverges to the right (grows unbounded almost surely)
- With $p < 1/2$ diverges to the left
- With $p = 1/2$ always come back to visit origin (almost surely)

- Because number of states is infinite we can have all states transient
  - They are not revisited after some time (more later)

2D Random Walk

- Take a step in random direction East, West, South or North

  $\Rightarrow$ E, W, S, N chosen with equal probability

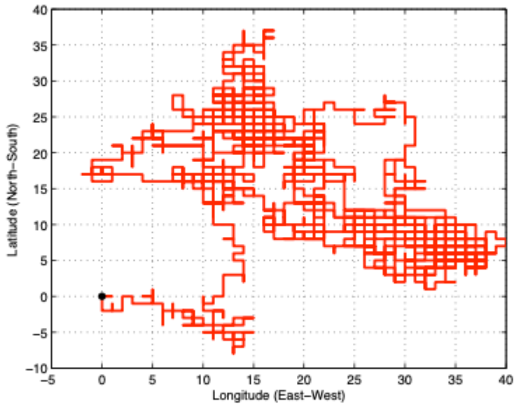
- States are pairs of coordinates (x, y)
  - $x = 0, \pm1, \pm2, \ldots$ and $y = 0, \pm1, \pm2, \ldots$
- Transiton probabilities are not zero only for points adjacent in the grid

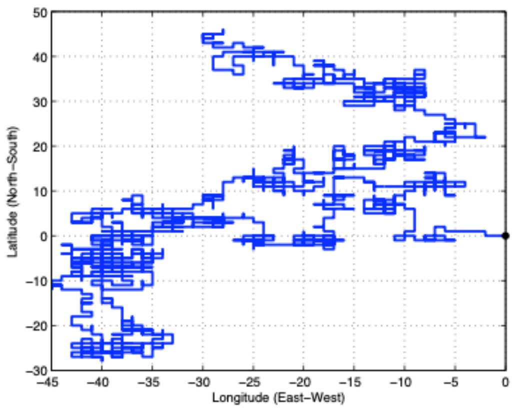
$$\mathrm{P}\left[x(t+1) = i+1, y(t+1) = j \,\middle|\, x(t) = i, y(t) = j\right] = \frac{1}{4}$$

$$\mathrm{P}\left[x(t+1) = i-1, y(t+1) = j \,\middle|\, x(t) = i, y(t) = j\right] = \frac{1}{4}$$

$$\mathrm{P}\left[x(t+1) = i, y(t+1) = j+1 \,\middle|\, x(t) = i, y(t) = j\right] = \frac{1}{4}$$

$$\mathrm{P}\left[x(t+1) = i, y(t+1) = j-1 \,\middle|\, x(t) = i, y(t) = j\right] = \frac{1}{4}$$





MutiStep Model

- What can be said about multiple transitions ?
- Transition probabilities between two time slots

$$P_{ij}^{2} := \mathrm{P}\left[X_{m+2} = j \,\middle|\, X_m = i\right]$$

- Probabilities of X_{m+n} given X_n $\Rightarrow$ n-step transition probabilities

$$P_{ij}^{n} := \mathrm{P}\left[X_{m+n} = j \,\middle|\, X_m = i\right]$$

- Relation between n-step, m-step and $(m+n)$-step transition probs.
  - Write P_{ij}^{m+n} in terms of P_{ij}^{m} and P_{ij}^{n}
- All questions answered by Chapman-Kolmogorov's equations

2^{nd} Step probabilities

- Start considering transition probs. between two time slots

$$P_{ij}^2 = \mathrm{P}\left[X_{n+2} = j \,\middle|\, X_n = i\right]$$

- Using the theorem of total probability

$$P_{ij}^2 = \sum_{k=1}^{\infty} \mathrm{P}\left[X_{n+2} = j \,\middle|\, X_{n+1} = k, X_n = i\right] \mathrm{P}\left[X_{n+1} = k \,\middle|\, X_n = i\right]$$

- In the first probability, conditioning on $X_n = i$ is unnecessary. Thus

$$P_{ij}^2 = \sum_{k=1}^{\infty} \mathrm{P}\left[X_{n+2} = j \,\middle|\, X_{n+1} = k\right] \mathrm{P}\left[X_{n+1} = k \,\middle|\, X_n = i\right]$$

- Which by definition yields

$$P_{ij}^2 = \sum_{k=1}^{\infty} P_{kj} P_{ik}$$

N+M step

- Identical argument can be made (condition on X_0 to simplify notation, possible because of time invariance)

$$P_{ij}^{m+n} = \mathrm{P}\left[X_{n+m} = j \,\middle|\, X_0 = i\right]$$

- Use theorem of total probability, remove unnecessary conditioning and use definitions of n-step and m-step transition probabilities

$$P_{ij}^{m+n} = \sum_{k=1}^{\infty} \mathrm{P}\left[X_{m+n} = j \,\middle|\, X_m = k, X_0 = i\right] \mathrm{P}\left[X_m = k \,\middle|\, X_0 = i\right]$$

$$P_{ij}^{m+n} = \sum_{k=1}^{\infty} \mathrm{P}\left[X_{m+n} = j \,\middle|\, X_m = k\right] \mathrm{P}\left[X_m = k \,\middle|\, X_0 = i\right]$$

$$P_{ij}^{m+n} = \sum_{k=1}^{\infty} P_{kj}^{n} P_{ik}^{m}$$

Equation

- Chapman Kolmogorov is intuitive. Recall

$$P_{ij}^{m+n} = \sum_{k=1}^{\infty} P_{kj}^{n} P_{ik}^{m}$$

- Between times 0 and $m + n$ time m occurred
- At time m, the chain is in some state $X_m = k$

  $\Rightarrow$ P_{ik}^m is the probability of going from $X_0 = i$ to $X_m = k$

  $\Rightarrow$ P_{kj}^n is the probability of going from $X_m = k$ to $X_{m+n} = j$

  $\Rightarrow$ Product $P_{ik}^m P_{kj}^n$ is then the probability of going from

  $X_0 = i$ to $X_{m+n} = j$ passing through $X_m = k$ at time m
- Since any k might have occurred sum over all k

Matrix equation

- Define matrices $\mathbf{P}^{(m)}$ with elements P_{ij}^{m}, $\mathbf{P}^{(n)}$ with elements P_{ij}^{n} and $\mathbf{P}^{(m+n)}$ with elements P_{ij}^{m+n}

- $\sum_{k=1}^{\infty} P_{kj}^{n} P_{ik}^{m}$ is the (i, j)-th element of matrix product $\mathbf{P}^{(m)}\mathbf{P}^{(n)}$
- Chapman Kolmogorov in matrix form

$$\mathbf{P}^{(m+n)} = \mathbf{P}^{(m)}\mathbf{P}^{(n)}$$

- Matrix of $(n+m)$-step transitions is product of n-step and m-step

N-th transition probabilities

- For $m = n = 1$ (2-step transition probabilities) matrix form is

$$\mathbf{P}^{(2)} = \mathbf{PP} = \mathbf{P}^2$$

- Proceed recursively backwards from n

$$\mathbf{P}^{(n)} = \mathbf{P}^{(n-1)}\mathbf{P} = \mathbf{P}^{(n-2)}\mathbf{PP} = \ldots = \mathbf{P}^n$$

- Have proved the following

Theorem

The matrix of n-step transition probabilities $P^{(n)}$ is given by the n-th power of the transition probability matrix $\mathbf{P}$. i.e.,

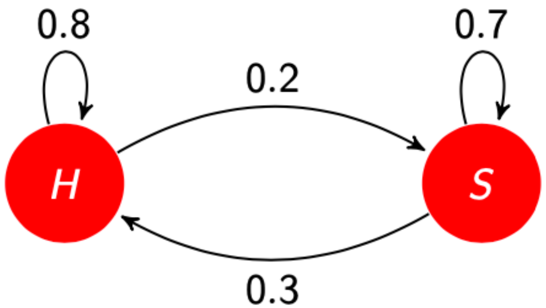
$$\mathbf{P}^{(n)} = \mathbf{P}^n$$

Henceforth we write $\mathbf{P}^n$

Happy Sad Game

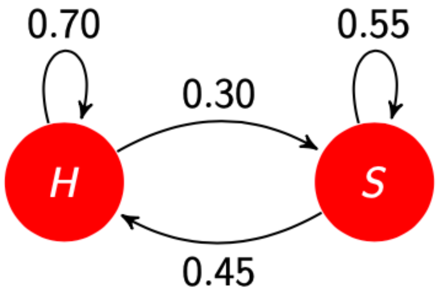
- Happiness transitions in one day (not the same as earlier example)

$$\mathbf{P} := \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}$$



- Transition probabilities between today and the day after tomorrow?

$$\mathbf{P}^2 := \begin{pmatrix} 0.70 & 0.30 \\ 0.45 & 0.55 \end{pmatrix}$$



A Chain

- ... After a week and after a month

$$\mathbf{P}^7 := \begin{pmatrix} 0.6031 & 0.3969 \\ 0.5953 & 0.4047 \end{pmatrix} \qquad \mathbf{P}^{30} := \begin{pmatrix} 0.6000 & 0.4000 \\ 0.6000 & 0.4000 \end{pmatrix}$$

- Matrices $\mathbf{P}^7$ and $\mathbf{P}^{30}$ almost identical $\Rightarrow \lim_{n\to\infty} \mathbf{P}^n$ exists
  - Note that this is a regular limit
- After a month transition from H to H with prob. 0.6 and from S to H also 0.6
- State becomes independent of initial condition
- Rationale: 1-step memory $\Rightarrow$ initial condition eventually forgotten

Unconditional probabilities

- All probabilities so far are conditional, i.e., $\mathrm{P}\left[X_n = j \,\middle|\, X_0 = i\right]$
- Want unconditional probabilities $p_j(n) := \mathrm{P}\left[X_n = j\right]$
- Requires specification of initial conditions $p_i(0) := \mathrm{P}\left[X_0 = i\right]$

- Using theorem of total probability and definitions of P_{ij}^n and $p_j(n)$

$$p_j(n) := \mathrm{P}\left[X_n = j\right] = \sum_{i=1}^{\infty} \mathrm{P}\left[X_n = j \,\middle|\, X_0 = i\right] \mathrm{P}\left[X_0 = i\right]$$

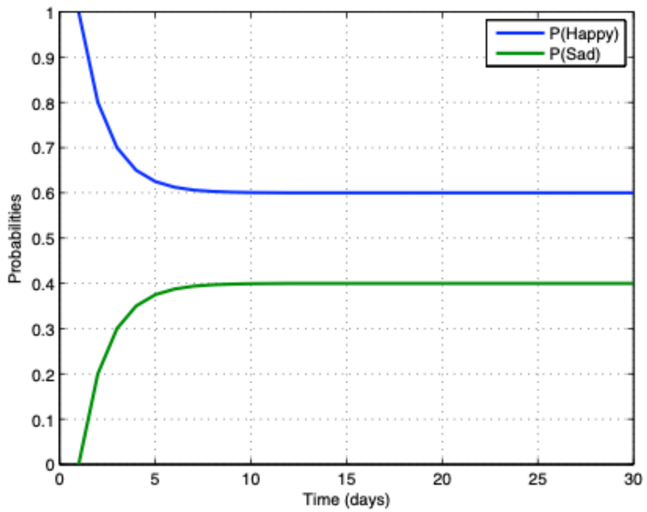
$$= \sum_{i=1}^{\infty} P_{ij}^n p_i(0)$$

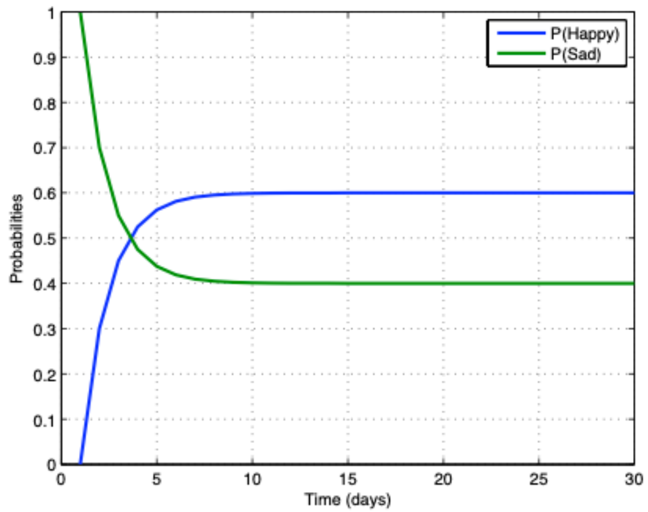
- Or in matrix form (define vector $\mathbf{p}(n) := [p_1(n), p_2(n), \ldots]^T$)

$$\mathbf{p}(n) = \mathbf{P}^{n\,T} \mathbf{p}(0)$$

- Transition probability matrix $\Rightarrow$ $\mathbf{P} := \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}$

$\mathbf{p}(0) = [1, 0]$ $\mathbf{p}(0) = [0, 1]$





- For large n probabilities $\mathbf{p}(t)$ are independent of initial state $\mathbf{p}(0)$

Markov Chain

- Stationary transition probability:
  - If ,for each i and j, P{ X_{t+1} = j | X_t = i } = P{ X_1 = j | X_0 = i }, for all t, then the transition probability are said to be stationary.

Markov Chain

- Steady-State Equations :

$$\pi_j = \sum_{i=0}^{M} \pi_i p_{ij} \quad \text{for i = 0, 1, …, M}$$

$$\sum_{j=0}^{M} \pi_j = 1$$

- , which consists of M+2 equations in M+1 unknowns.

Markov Chain

- A stochastic process $\{X_t\}$ is a Markov chain if it has Markovian property.

- Markovian property:
  - $P\{ X_{t+1} = j \mid X_0 = k_0, X_1 = k_1, \ldots, X_{t-1} = k_{t-1}, X_t = i \}$
    $= P\{ X_{t+1} = j \mid X_t = i \}$

- $P\{ X_{t+1} = j \mid X_t = i \}$ is called the transition probability.