Problems classification: C^∞-problem

C^∞-problem

$$f(x) \to \min,$$

$$x \in X.$$

Question: will the situation be better if we assume $f \in C^k$, $k \geq 1$?

Answer: No!!!

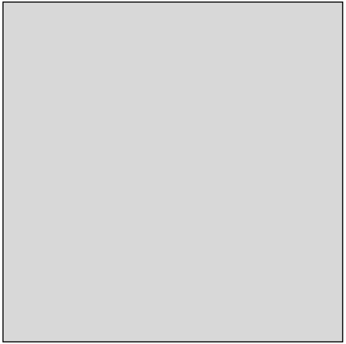
Whitney theorem. *Let $A \subset \mathbb{R}^n$ be a closed set. Then there exists a function $h \in C^\infty$ such that*

$$A = \{x \in \mathbb{R}^n : h(x) = 0\}.$$

$$\Downarrow$$

$$h^2(x) \to \min.$$

- Serpinsky carpet



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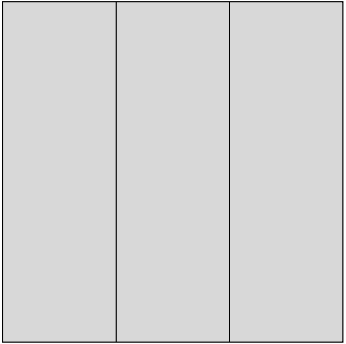

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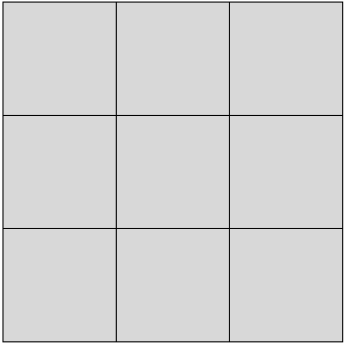

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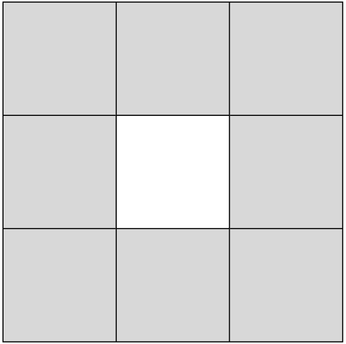

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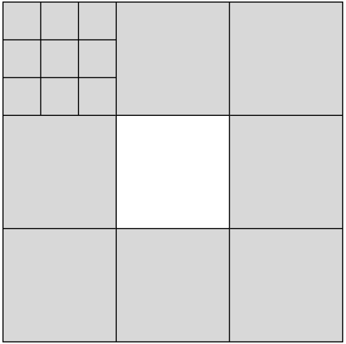

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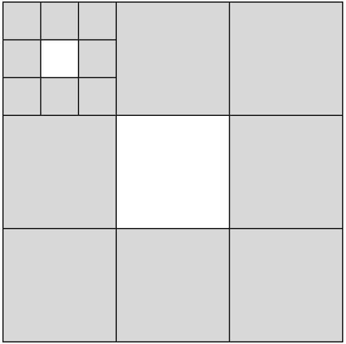

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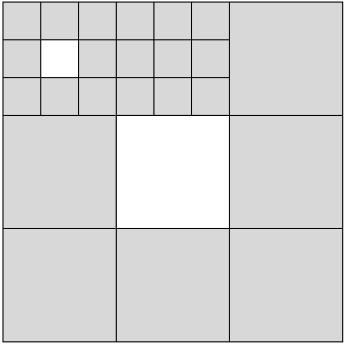

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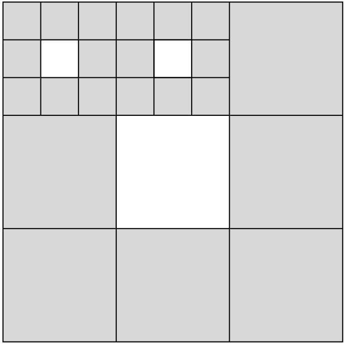

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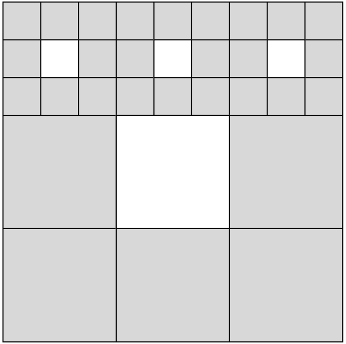

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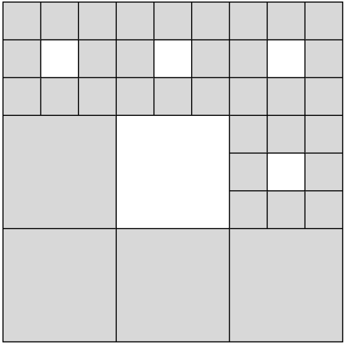

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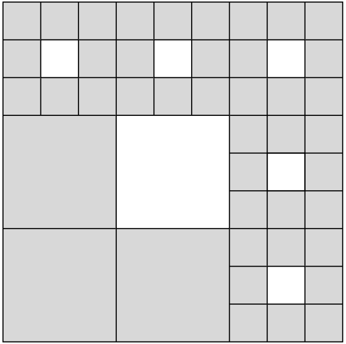

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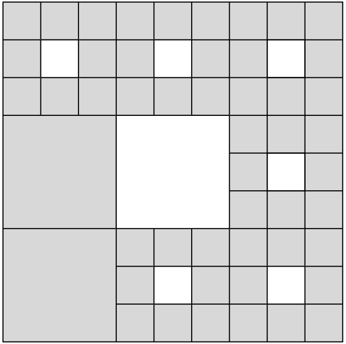

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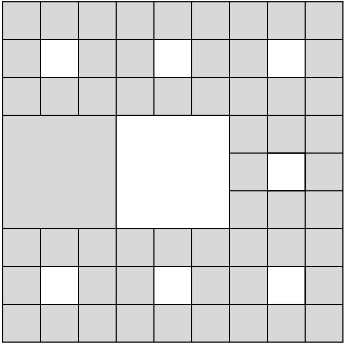

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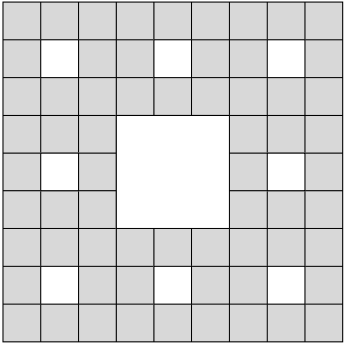

Problems classification: C^∞-problem

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$$\Downarrow$$

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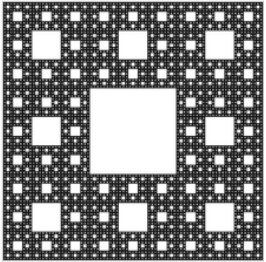
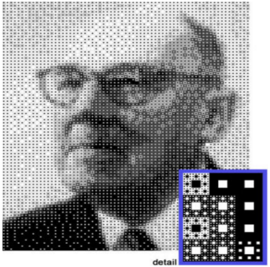


Serpinsky carpet



V. Serpinsky

- Differentiability of any order is a very nice property, however applicable for local analysis like stationarity, stability or sensitivity.
- We need an assumption in addition to differentiability!

- Problem 1. For given function $f(x) = |x^2 - 1| + 0.5x - 8$ find all points of local minimum and select global minimum points. (0.3)

- *Solution.*

$$f(x) = \begin{cases} f_1(x) = x^2 - 1 + 0.5x - 8 = x^2 + 0.5x - 9, & x^2 - 1 \geqslant 0; \\ f_2(x) = -x^2 + 1 + 0.5x - 8 = -x^2 + 0.5x - 7, & x^2 - 1 \leqslant 0. \end{cases}$$

$$x^2 - 1 = 0 \;\Leftrightarrow\; (x-1)(x+1) = 0 \;\Leftrightarrow\; (x = -1) \bigvee (x = 1).$$

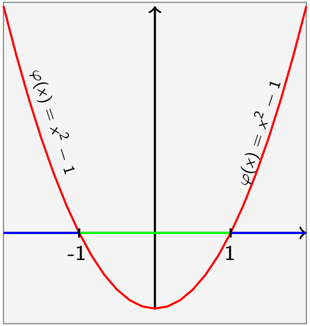
$$f_1(-1) = f_2(-1) = -8.5 \text{ and } f_1(1) = f_2(1) = -7.5$$

$$\Downarrow$$

$$\underline{x^2 - 1 \leqslant 0 \;\Leftrightarrow\; x \in [-1, 1]}, \quad \underline{x^2 - 1 \geqslant 0 \;\Leftrightarrow\; x \notin (-1, 1)}.$$

$$\Downarrow$$

$$f(x) = \begin{cases} f_1(x) = x^2 + 0.5x - 9, & x \leqslant -1; \\ f_2(x) = -x^2 + 0.5x - 7, & -1 \leqslant x \leqslant 1; \\ f_1(x) = x^2 + 0.5x - 9, & x \geqslant 1. \end{cases}$$



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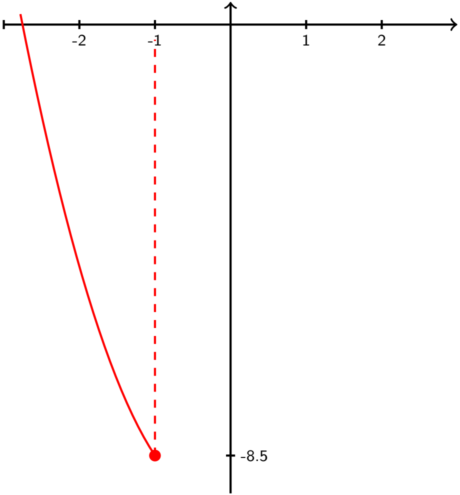


- *Solution.*

$$f(x) = \begin{cases} f_1(x) = x^2 + 0.5x - 9, & x \leqslant -1; \\ f_2(x) = -x^2 + 0.5x - 7, & -1 \leqslant x \leqslant 1; \\ f_1(x) = x^2 + 0.5x - 9, & x \geqslant 1. \end{cases}$$

$$\min f(x) \;\; s.t. \;\; x \in \mathbb{R}.$$

<u>*Subproblem A.*</u>

$$\min\{f_1(x) = x^2 + 0.5x - 9\} \;\; s.t. \;\; x \leqslant -1.$$

$$f_1'(x) = 2x + 0.5 < 0, \;\; x \leqslant -1;$$

f_1 is decreasing when $x \leqslant -1$;

$$x^{*,1} = \text{argmin}\{f_1(x) : x \leqslant -1\} = -1.$$

- Problem 1. For given function $f(x) = |x^2 - 1| + 0.5x - 8$ find all points of local minimum and select global minimum points.

- *Solution.*

$$f(x) = \begin{cases} f_1(x) = x^2 + 0.5x - 9, & x \leqslant -1; \\ f_2(x) = -x^2 + 0.5x - 7, & -1 \leqslant x \leqslant 1; \\ f_1(x) = x^2 + 0.5x - 9, & x \geqslant 1. \end{cases}$$

$$\min f(x) \;\; s.t. \;\; x \in \mathbb{R}.$$

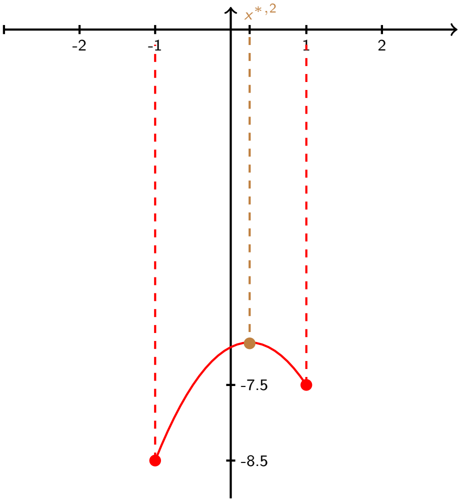
<u>*Subproblem B.*</u>

$$\min\{f_2(x) = -x^2 + 0.5x - 7\} \;\; s.t. \;\; x \in [-1, 1].$$

$$f_2'(x) = -2x + 0.5 = 0 \;\Rightarrow\; x^{*,2} = 0.25, \;\; f_2''(x) = -2 < 0;$$

$$f_2(-1) = -8.5, \;\; f_2(1) = -7.5;$$

$$x^{*,3} = \text{argmin}\{f_2(-1), f_2(1)\} = -1 = x^{*,1}.$$



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$$\min f(x) \ \ s.t. \ \ x \in \mathbb{R}.$$

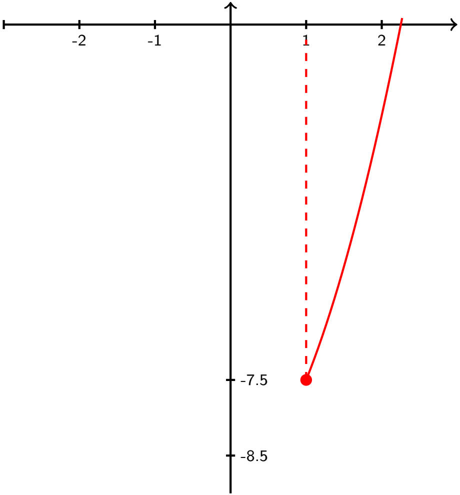
Subproblem C.

$$\min\{f_1(x) = x^2 + 0.5x - 9\} \ \ s.t. \ \ x \geqslant 1.$$

$$f_1'(x) = 2x + 0.5 > 0, \quad x \geqslant 1;$$

f_1 is increasing when $x \geqslant 1$;

$$x^{*,4} = \text{argmin}\{f_1(x) : x \geqslant 1\} = 1.$$



- Problem 1. For given function $f(x) = |x^2 - 1| + 0.5x - 8$ find all points of local minimum and select global minimum points.

- *Solution.*

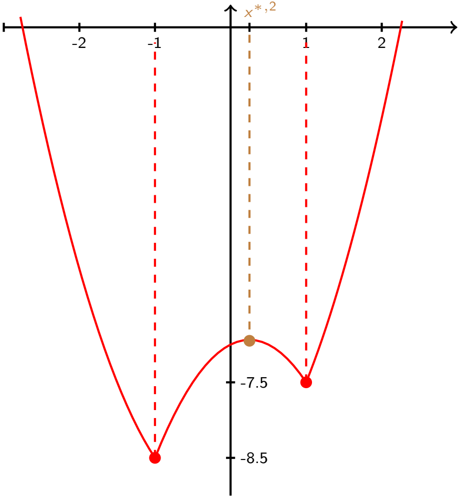
$$f(x) = \begin{cases} f_1(x) = x^2 + 0.5x - 9, & x \leqslant -1; \\ f_2(x) = -x^2 + 0.5x - 7, & -1 \leqslant x \leqslant 1; \\ f_1(x) = x^2 + 0.5x - 9, & x \geqslant 1. \end{cases}$$

$$\min f(x) \ \ s.t. \ \ x \in \mathbb{R}.$$

$x^{*,2} = 0.25$ — loc.max.;

$x^{*,4} = 1, \ f(x^{*,4}) = -7.5$ — loc.min.;

$x^{*,1} = x^{*,3} = -1, \ f(x^{*,1}) = -8.5$ — glob.min.



- Standard optimality conditions. For given function $f(x_1, x_2, x_3) = x_1 x_2^2 x_3^3 (1 - x_1 - 2x_2 - 3x_3)$

  a. find all stationary points;

  b. check the second order optimality conditions and classify the found stationary points.

- Solution a.

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \\ \frac{\partial f(x)}{\partial x_3} \end{pmatrix} = \begin{pmatrix} x_2^2 x_3^3 (1 - x_1 - 2x_2 - 3x_3) - x_1 x_2^2 x3^3 \\ 2x_1 x_2 x_3^3 (1 - x_1 - 2x_2 - 3x_3) - 2x_1 x_2^2 x_3^3 \\ 3x_1 x_2^2 x_3^2 (1 - x_1 - 2x_2 - 3x_3) - 3x_1 x_2^2 x_3^3 \end{pmatrix} = \begin{pmatrix} x_2^2 x_3^3 (1 - 2x_1 - 2x_2 - 3x_3) \\ 2x_1 x_2 x_3^3 (1 - x_1 - 3x_2 - 3x_3) \\ 3x_1 x_2^2 x_3^2 (1 - x_1 - 2x_2 - 4x_3) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Stationary point $x^{*,1}$ corresponds to the solution of linear system

$$\begin{aligned} 1 - 2x_1 - 2x_2 - 3x_3 &= 0, \\ 1 - x_1 - 3x_2 - 3x_3 &= 0, \\ 1 - x_1 - 2x_2 - 4x_3 &= 0. \end{aligned} \qquad \Longrightarrow \qquad x^{*,1} = \begin{pmatrix} \frac{1}{7} \\ \frac{1}{7} \\ \frac{1}{7} \end{pmatrix}.$$

- Other stationary points satisfy nonlinear system

$$\begin{aligned} x_2^2 x_3^3 &= 0, \\ 2x_1 x_2 x_3^3 &= 0, \\ 3x_1 x_2^2 x_3^2 &= 0. \end{aligned} \quad \Longrightarrow \quad x^{*,2},\ x^{*,3}, x^{*,4}.$$

$$x^{*,2} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}, x_1,\ x_2 \text{ are arbitrary.}$$

$$x^{*,3} = \begin{pmatrix} 0 \\ \frac{1-3x_3}{2} \\ x_3 \end{pmatrix}, x_3 \text{ is arbitrary;}$$

$$x^{*,4} = \begin{pmatrix} x_1 \\ 0 \\ x_3 \end{pmatrix}, x_1,\ x_3 \text{ are arbitrary;}$$

- Solution b.

$$H(x) = \begin{pmatrix} -2x_2^2 x_3^3 & -2x_2 x_3^3(2x_1 + 3x_2 + 3x_3 - 1) & -6(x_1 + x_2 + 2x_3 - 0.5)x_3^2 x_2^2 \\ -2x_2 x_3^3(2x_1 + 3x_2 + 3x_3 - 1) & -2x_1 x_3^3(x_1 + 6x_2 + 3x_3 - 1) & -6x_1 x_2 x_3^2(x_1 + 3x_2 + 4x_3 - 1) \\ -6(x_1 + x_2 + 2x_3 - 0.5)x_3^2 x_2^2 & -6x_1 x_2 x_3^2(x_1 + 3x_2 + 4x_3 - 1) & -6x_1 x_2^2 x_3(x_1 + 2x_2 + 6x_3 - 1) \end{pmatrix}$$

$$H(x^{*,1}) = \begin{pmatrix} -0.0001189980367 & -0.0001189980368 & -0.0001784970555 \\ -0.0001189980368 & -0.0003569941104 & -0.0003569941105 \\ -0.0001784970555 & -0.0003569941105 & -0.0007139882208 \end{pmatrix}$$

$$\lambda(x^{*,1}) = \begin{pmatrix} -0.000987550538936726 \\ -0.0000653500619736083 \\ -0.00013707976698966 \end{pmatrix} \implies \quad H(x^{*,1}) \prec 0, \text{ (ND)}, x^{*,1} \text{ is a local maximum.}$$

- $H(x^{*,2})$, $H(x^{*,3})$, $H(x^{*,4})$ are indefinite (each contains an eigenvalue equal to zero). Therefore, no more conclusion can be made on $x^{*,2}$, $x^{*,3}$, $x^{*,4}$ (on the base of the first and second derivatives).