- Unconstrained quadratic optimization: $\min_{x \in \mathbb{R}^n} \{f(x) = x^\top Q x + c^\top x\}$

- Algebraically based algorithm.

  Step 1. Calculate an eigenvalue decomposition: $Q \sim (\Lambda, V)$;

  Step 2. Determine index sets

  $$\mathcal{I}^- = \{i : \lambda_i < 0\}, \mathcal{I}^0 = \{i : \lambda_i = 0\}, \mathcal{I}^+ = \{i : \lambda_i > 0\};$$

  Step 3. Determine integers $n^- = |\mathcal{I}^-|, n^0 = |\mathcal{I}^0|, n^+ = |\mathcal{I}^+|$;

  Step 4. If $n^0 > 0$ then calculate values $\gamma_i = v_i^\top c, i \in \mathcal{I}^0$;

  If $\gamma_\xi \neq 0$ for some $\xi \in \mathcal{I}^0$ then stop: there are no stationary points;

  Otherwise goto Step 5;

  Step 5. Calculate diagonal matrix $\Theta = V^\top V$;

  Step 6. Check the below cases, perform the corresponding computations and stop:

Case 6.1. $(n^+ > 0)\&(n^0 = 0)\&(n^- = 0)$: calculate the unique minimum

$$x^* = -\frac{1}{2} V \Theta^{-1} \Lambda^{-1} V^\top c,$$

minimal value $f^* = -\frac{1}{4} \sum_{i \in \mathcal{I}^+} \frac{1}{\lambda_i \theta_i} (v_i^\top c)^2$;

Case 6.2. $(n^+ > 0)\&(n^0 > 0)\&(n^- = 0)$: calculate $\tau_i^* = -\frac{1}{2\lambda_i \theta_i} v_i^\top c, i \in \mathcal{I}^+$,

determine the set of multiple minima

$$X^* = \{x : x = \sum_{i \in \mathcal{I}^+} \tau_i^* v_i + \sum_{i \in \mathcal{I}^0} \tau_i v_i, \tau_i \in \mathbb{R}, i \in \mathcal{I}^0\},$$

calculate a representative minimum $x^* \in X^*$ corresponding to $\tau_i = 0, i \in \mathcal{I}^0$:

$$x^* = \sum_{i \in \mathcal{I}^+} \tau_i^* v_i,$$

calculate minimal value $f^* = -\frac{1}{4} \sum_{i \in \mathcal{I}^+} \frac{1}{\lambda_i \theta_i} (v_i^\top c)^2$;

Case 6.3. $(n^+ > 0)\&(n^0 = 0)\&(n^- > 0)$: calculate $\tau_i^* = -\frac{1}{2\lambda_i\theta_i} v_i^\top c, i \in \mathcal{I}^+ \cup \mathcal{I}^-$,

unique stationary (saddle) point $x^* = \sum_{i \in \mathcal{I}^+ \cup \mathcal{I}^-} \tau_i^* v_i$,

stationary value $f^* = -\frac{1}{4} \sum_{i \in \mathcal{I}^+ \cup \mathcal{I}^-} \frac{1}{\lambda_i\theta_i} (v_i^\top c)^2$;

Case 6.4. $(n^+ > 0)\&(n^0 > 0)\&(n^- > 0)$: calculate $\tau_i^* = -\frac{1}{2\lambda_i\theta_i} v_i^\top c, i \in \mathcal{I}^+ \cup \mathcal{I}^-$,

determine the set of multiple stationary (saddle) points

$$X^* = \{x : x = \sum_{i \in \mathcal{I}^+ \cup \mathcal{I}^-} \tau_i^* v_i + \sum_{i \in \mathcal{I}^0} \tau_i v_i, \tau_i \in \mathbb{R}, i \in \mathcal{I}^0\},$$

calculate a representative stationary point $x^* \in X^*$ $(\tau_i = 0, i \in \mathcal{I}^0)$:

$$x^* = \sum_{i \in \mathcal{I}^+ \cup \mathcal{I}^-} \tau_i^* v_i,$$

calculate stationary value $f^* = -\frac{1}{4} \sum_{i \in \mathcal{I}^+ \cup \mathcal{I}^-} \frac{1}{\lambda_i\theta_i} (v_i^\top c)^2$;

Case 6.5. $(n^+ = 0)\&(n^0 > 0)\&(n^- > 0)$: calculate $\tau_i^* = -\frac{1}{2\lambda_i\theta_i} v_i^\top c, i \in \mathcal{I}^-$,

determine the set of multiple maxima

$$X^* = \{x : x = \sum_{i\in\mathcal{I}^-} \tau_i^* v_i + \sum_{i\in\mathcal{I}^0} \tau_i v_i, \tau_i \in \mathbb{R}, i \in \mathcal{I}^0\},$$

calculate a representative maximum $x^* \in X^*$ corresponding to $\tau_i = 0, i \in \mathcal{I}^0$:

$$x^* = \sum_{i\in\mathcal{I}^-} \tau_i^* v_i,$$

calculate maximal value $f^* = -\frac{1}{4} \sum_{i\in\mathcal{I}^-} \frac{1}{\lambda_i\theta_i}(v_i^\top c)^2$;

Case 6.6. $(n^+ = 0)\&(n^0 = 0)\&(n^- > 0)$: calculate the unique maximum

$$x^* = -\frac{1}{2} V\Theta^{-1}\Lambda^{-1}V^\top c,$$

calculate maximal value $f^* = -\frac{1}{4} \sum_{i\in\mathcal{I}^-} \frac{1}{\lambda_i\theta_i}(v_i^\top c)^2$.

- Representing $f(x) = x^\top Qx + c^\top x$ as a weighted sum of squares

Step 1. Calculate an eigenvalue decomposition: $Q \sim (\Lambda, V)$;

Step 2. Determine index sets

$$\mathcal{I}^- = \{i : \lambda_i < 0\}, \mathcal{I}^0 = \{i : \lambda_i = 0\}, \mathcal{I}^+ = \{i : \lambda_i > 0\};$$

Step 3. Calculate parameters $\gamma_i = v_i^\top c,\ i = 1, \ldots, n$;

Step 4. Calculate parameters $\theta_i = v_i^\top v_i$, i=1,. . . ,n;

Step 5. Define the representation:

$$f(x) = \sum_{i\in\mathcal{I}^-} \frac{\lambda_i}{\theta_i}\left(v_i^\top x + \frac{1}{2\lambda_i}\gamma_i\right)^2 + \sum_{i\in\mathcal{I}^0} \frac{1}{\theta_i}\gamma_i \cdot v_i^\top x + \\ + \sum_{i\in\mathcal{I}^+} \frac{\lambda_i}{\theta_i}\left(v_i^\top x + \frac{1}{2\lambda_i}\gamma_i\right)^2 - \frac{1}{4} \sum_{i\in\mathcal{I}^-\cup\mathcal{I}^+} \frac{1}{\lambda_i\theta_i}\gamma_i^2.$$

Example 1. $f(x_1, x_2) = -57x_1^2 + 12x_1x_2 - 48x_2^2 + 50x_1 - 20x_2 \Rightarrow Q = \begin{pmatrix} -57 & 6 \\ 6 & -48 \end{pmatrix}, c = \begin{pmatrix} 50 \\ -20 \end{pmatrix};$

$$f(x) = \underbrace{\begin{pmatrix} x_1 & x_2 \end{pmatrix}}_{x^\top} \underbrace{\begin{pmatrix} -57 & 6 \\ 6 & -48 \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} 50 & -20 \end{pmatrix}}_{c^\top} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x}$$

Step 1. Calculate an eigenvalue decomposition: $Q \sim (\Lambda, V)$,

$$\Lambda = \begin{pmatrix} -60 & 0 \\ 0 & -45 \end{pmatrix}, \begin{matrix} \lambda_1 = -60, \\ \lambda_2 = -45, \end{matrix} \; V = \begin{pmatrix} -2 & \frac{1}{2} \\ 1 & 1 \end{pmatrix}, \; v_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix};$$

Step 2. Determine index sets $\mathcal{I}^- = \{1, 2\}$, $\mathcal{I}^0 = \emptyset$, $\mathcal{I}^+ = \emptyset$;

Step 3. Determine integers $n^- = |\mathcal{I}^-| = 2$, $n^0 = |\mathcal{I}^0| = 0$, $n^+ = |\mathcal{I}^+| = 0$;

Step 4. Since $n^0 = 0$, goto Step 5;

Step 5. Calculate diagonal matrix

$$\Theta = V^\top V = \begin{pmatrix} -2 & 1 \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} -2 & \frac{1}{2} \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & \frac{5}{4} \end{pmatrix};$$

Step 6. Case 6.6. $(n^+ = 0)\&(n^0 = 0)\&(n^- > 0)$: calculate the unique maximum

$$x^* = -\frac{1}{2} V \Theta^{-1} \Lambda^{-1} V^\top c =$$

$$= -\frac{1}{2} \cdot \underbrace{\begin{pmatrix} -2 & \frac{1}{2} \\ 1 & 1 \end{pmatrix}}_{V} \underbrace{\begin{pmatrix} \frac{1}{5} & 0 \\ 0 & \frac{4}{5} \end{pmatrix}}_{\Theta^{-1}} \underbrace{\begin{pmatrix} -\frac{1}{60} & 0 \\ 0 & -\frac{1}{45} \end{pmatrix}}_{\Lambda^{-1}} \underbrace{\begin{pmatrix} -2 & 1 \\ \frac{1}{2} & 1 \end{pmatrix}}_{V^\top} \underbrace{\begin{pmatrix} 50 \\ -20 \end{pmatrix}}_{c} =$$

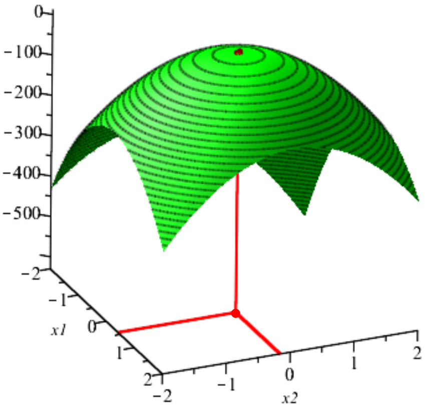
$$= \begin{pmatrix} \frac{19}{45} \\ -\frac{7}{45} \end{pmatrix} \approx \begin{pmatrix} 0.422 \\ -0.155 \end{pmatrix},$$

calculate maximal value $\quad f^* = -\frac{1}{4} \sum_{i \in \mathcal{I}^-} \frac{1}{\lambda_i \theta_i} (v_i^\top c)^2 =$

$$= -\frac{1}{4} \left[\frac{1}{\underbrace{-60}_{\lambda_1} \cdot \underbrace{5}_{\theta_1}} \cdot \left(\underbrace{\begin{pmatrix} -2 & 1 \end{pmatrix}}_{v_1^\top} \underbrace{\begin{pmatrix} 50 \\ -20 \end{pmatrix}}_{c} \right)^2 + \frac{1}{\underbrace{-45}_{\lambda_2} \cdot \underbrace{\left(\frac{5}{4}\right)}_{\theta_2}} \cdot \left(\underbrace{\begin{pmatrix} \frac{1}{2} & 1 \end{pmatrix}}_{v_2^\top} \underbrace{\begin{pmatrix} 50 \\ -20 \end{pmatrix}}_{c} \right)^2 \right] =$$

$$= -\frac{1}{4} \left[\frac{1}{300} \cdot 120^2 - \frac{4}{45 \cdot 5} \cdot 5^2 \right] = \frac{109}{9} \approx 12.111.$$

- Geometrical interpretation



$$f(x) = -57x_1^2 + 12x_1x_2 - 48x_2^2 + 50x_1 - 20x_2,$$

unique maximum $x^* = \begin{pmatrix} \frac{19}{45} \\ -\frac{7}{45} \end{pmatrix} \approx \begin{pmatrix} 0.422 \\ -0.155 \end{pmatrix}$,

maximal value $f^* = \dfrac{109}{9} \approx 12.111.$

- Representing $f(x) = -57x_1^2 + 12x_1x_2 - 48x_2^2 + 50x_1 - 20x_2$ as a weighted sum of squares

$$f(x) = -12(-2x_1 + x_2 + 1)^2 - 36\left(\frac{1}{2}x_1 + x_2 - \frac{1}{18}\right)^2 + \frac{109}{9}.$$

End of Example 1

Example 2. $f(x_1, x_2) = 6x_1^2 - 12x_1x_2 - 3x_2^2 - 8x_1 + 24x_2 \Rightarrow Q = \begin{pmatrix} 6 & -6 \\ -6 & -3 \end{pmatrix}, c = \begin{pmatrix} -8 \\ 24 \end{pmatrix};$

$$f(x) = \underbrace{\begin{pmatrix} x_1 & x_2 \end{pmatrix}}_{x^\top} \underbrace{\begin{pmatrix} 6 & -6 \\ -6 & -3 \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} -8 & 24 \end{pmatrix}}_{c^\top} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x}$$

Step 1. Calculate an eigenvalue decomposition: $Q \sim (\Lambda, V)$,

$$\Lambda = \begin{pmatrix} -6 & 0 \\ 0 & 9 \end{pmatrix}, \quad \begin{matrix} \lambda_1 = -6, \\ \lambda_2 = 9, \end{matrix} \quad V = \begin{pmatrix} \frac{1}{2} & -2 \\ 1 & 1 \end{pmatrix}, v_1 = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix};$$

Step 2. Determine index sets $\mathcal{I}^- = \{1\}$, $\mathcal{I}^0 = \emptyset$, $\mathcal{I}^+ = \{2\}$;

Step 3. Determine integers $n^- = |\mathcal{I}^-| = 1$, $n^0 = |\mathcal{I}^0| = 0$, $n^+ = |\mathcal{I}^+| = 1$;

Step 4. Since $n^0 = 0$, goto Step 5;

Step 5. Calculate diagonal matrix

$$\Theta = V^\top V = \begin{pmatrix} \frac{1}{2} & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{4} & 0 \\ 0 & 5 \end{pmatrix};$$

Step 6. Case 6.3. $(n^+ > 0) \& (n^0 = 0) \& (n^- > 0)$: unique stationary (saddle) point

$$x^* = -\frac{1}{2} V \Theta^{-1} \Lambda^{-1} V^\top c =$$

$$= -\frac{1}{2} \cdot \underbrace{\begin{pmatrix} \frac{1}{2} & -2 \\ 1 & 1 \end{pmatrix}}_{V} \underbrace{\begin{pmatrix} \frac{4}{5} & 0 \\ 0 & \frac{1}{5} \end{pmatrix}}_{\Theta^{-1}} \underbrace{\begin{pmatrix} -\frac{1}{6} & 0 \\ 0 & \frac{1}{9} \end{pmatrix}}_{\Lambda^{-1}} \underbrace{\begin{pmatrix} \frac{1}{2} & 1 \\ -2 & 1 \end{pmatrix}}_{V^\top} \underbrace{\begin{pmatrix} -8 \\ 24 \end{pmatrix}}_{c} =$$

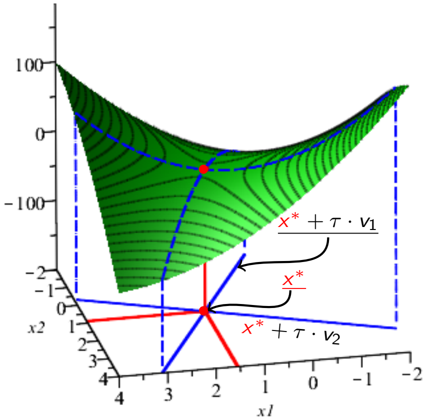
$$= \begin{pmatrix} \frac{14}{9} \\ \frac{8}{9} \end{pmatrix} \approx \begin{pmatrix} 1.556 \\ 0.889 \end{pmatrix},$$

calculate stationary (saddle) value $f^* = -\frac{1}{4}\sum_{i\in\mathcal{I}^+\cup\mathcal{I}^-}\frac{1}{\lambda_i\theta_i}(v_i^\top c)^2 =$

$$= -\frac{1}{4}\left[\frac{1}{\underbrace{-6}_{\lambda_1}\cdot\underbrace{\left(\frac{5}{4}\right)}_{\theta_1}}\cdot\left(\underbrace{\begin{pmatrix}\frac{1}{2} & 1\end{pmatrix}}_{v_1^\top}\underbrace{\begin{pmatrix}-8\\24\end{pmatrix}}_{c}\right)^2 + \frac{1}{\underbrace{9}_{\lambda_2}\cdot\underbrace{5}_{\theta_2}}\cdot\left(\underbrace{\begin{pmatrix}-2 & 1\end{pmatrix}}_{v_2^\top}\underbrace{\begin{pmatrix}-8\\24\end{pmatrix}}_{c}\right)^2\right] =$$

$$= -\frac{1}{4}\left[-\frac{2}{15}\cdot 20^2 + \frac{1}{45\cdot 5}\cdot 40^2\right] = \frac{40}{9} \approx 4.444.$$

- Geometrical interpretation:



$f(x) = 6x_1^2 - 12x_1x_2 - 3x_2^2 - 8x_1 + 24x_2,$

unique stationary (saddle) point maximum

$$x^* = \begin{pmatrix} \frac{14}{9} \\ \frac{8}{9} \end{pmatrix} \approx \begin{pmatrix} 1.556 \\ 0.889 \end{pmatrix},$$

stationary value $f^* = \dfrac{40}{9} \approx 4.444.$

$$\varphi_1(\tau) = f(x^* + \tau \cdot v_1) = \frac{40}{9} - \frac{15}{2}\tau^2$$

φ_1 is concave

$$\varphi_2(\tau) = f(x^* + \tau \cdot v_2) = \frac{40}{9} + 45\tau^2$$

φ_2 is convex

- Representing $f(x) = 6x_1^2 - 12x_1x_2 - 3x_2^2 - 8x_1 + 24x_2$ as a weighted sum of squares

$$f(x) = -\frac{24}{5}\left(\frac{1}{2}x_1 + x_2 - \frac{5}{3}\right)^2 + \frac{9}{5}\left(-2x_1 + x_2 + \frac{20}{9}\right)^2 + \frac{40}{9}.$$

End of Example 2

Example 3. $f(x_1, x_2) = 4x_1^2 + 4x_1x_2 + x_2^2 + 50x_1 - 45x_2 \Rightarrow Q = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}, c = \begin{pmatrix} 50 \\ -45 \end{pmatrix};$

$$f(x) = \underbrace{\begin{pmatrix} x_1 & x_2 \end{pmatrix}}_{x^\top} \underbrace{\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} 50 & -45 \end{pmatrix}}_{c^\top} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x}$$

Step 1. Calculate an eigenvalue decomposition: $Q \sim (\Lambda, V)$,

$$\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}, \quad \begin{matrix} \lambda_1 = 0, \\ \lambda_2 = 5, \end{matrix} \quad V = \begin{pmatrix} -\frac{1}{2} & 2 \\ 1 & 1 \end{pmatrix}, v_1 = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix};$$

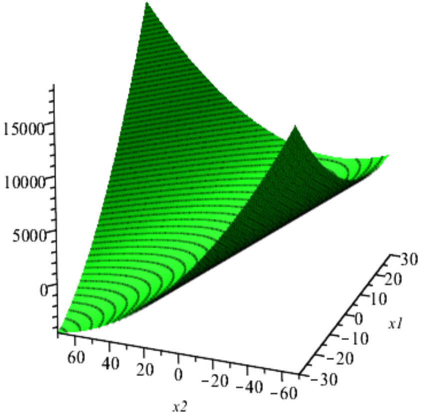
Step 2. Determine index sets $\mathcal{I}^- = \emptyset$, $\mathcal{I}^0 = \{1\}$, $\mathcal{I}^+ = \{2\}$;

Step 3. Determine integers $n^- = |\mathcal{I}^-| = 0$, $n^0 = |\mathcal{I}^0| = 1$, $n^+ = |\mathcal{I}^+| = 1$;

Step 4. $n^0 = 1 > 0 \Rightarrow$ calculate $\gamma_1 = v_1^\top c = \dfrac{1}{2} \cdot (-8) + 1 \cdot 24 = 20 \neq 0 \Rightarrow$

$\Rightarrow$ no stationary points: stop.

- Geometrical interpretation



$$f(x) = 4x_1^2 + 4x_1x_2 + x_2^2 + 50x_1 - 45x_2,$$

no stationary points

- Representing $f(x) = 4x_1^2 + 4x_1x_2 + x_2^2 + 50x_1 - 45x_2$ as a weighted sum of squares and an affine function

$$f(x) = \left(2x_1 + x_2 + \frac{11}{2}\right)^2 + 28x_1 - 56x_2 + \frac{121}{4}.$$

End of Example 3

Example 4. $f(x_1,x_2) = 4x_1^2 - 12x_1x_2 + 9x_2^2 + 4x_1 - 6x_2 \Rightarrow Q = \begin{pmatrix} 4 & -6 \\ -6 & 9 \end{pmatrix}, c = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$;

$$f(x) = \underbrace{\begin{pmatrix} x_1 & x_2 \end{pmatrix}}_{x^\top} \underbrace{\begin{pmatrix} 4 & -6 \\ -6 & 9 \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} 4 & -6 \end{pmatrix}}_{c^\top} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x}$$

Step 1. Calculate an eigenvalue decomposition: $Q \sim (\Lambda, V)$,

$$\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & 13 \end{pmatrix}, \quad \begin{aligned} \lambda_1 &= 0, \\ \lambda_2 &= 13, \end{aligned} \quad V = \begin{pmatrix} \frac{3}{2} & -\frac{2}{3} \\ 1 & 1 \end{pmatrix}, v_1 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix};$$

Step 2. Determine index sets $\mathcal{I}^- = \emptyset,\ \mathcal{I}^0 = \{1\},\ \mathcal{I}^+ = \{2\}$;

Step 3. Determine integers $n^- = |\mathcal{I}^-| = 0,\ n^0 = |\mathcal{I}^0| = 1,\ n^+ = |\mathcal{I}^+| = 1$;

Step 4. $n^0 = 1 > 0 \ \Rightarrow$ calculate $\gamma_1 = v_1^\top c = \frac{3}{2} \cdot 4 + 1 \cdot (-6) = 0 \ \Rightarrow$ goto Step 5;

Step 5. Calculate diagonal matrix

$$\Theta = V^\top V = \begin{pmatrix} \frac{3}{2} & 1 \\ -\frac{2}{3} & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{2} & -\frac{2}{3} \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{13}{4} & 0 \\ 0 & \frac{13}{9} \end{pmatrix};$$

Step 6. Case 6.2. $(n^+ > 0)\&(n^0 > 0)\&(n^- = 0)$: multiple minima

calculate $\tau_i^* = -\dfrac{1}{2\lambda_i\theta_i} v_i^\top c, i \in \mathcal{I}^+$, $\tau_2^* = -\frac{1}{2\cdot 13\cdot\left(\frac{13}{9}\right)} \cdot \underbrace{\begin{pmatrix} -\frac{2}{3} & 1 \end{pmatrix}}_{v_2^\top} \underbrace{\begin{pmatrix} 4 \\ -6 \end{pmatrix}}_{c} = \frac{3}{13}$

determine the set of multiple minima

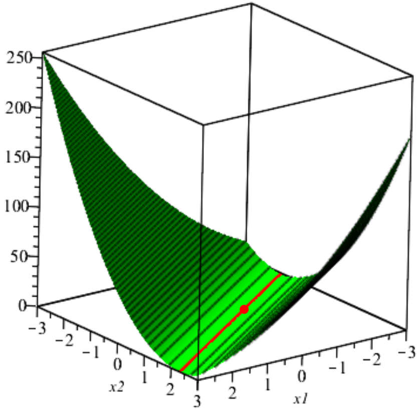
$$X^* = \{x : x = \sum_{i\in\mathcal{I}^+} \tau_i^* v_i + \sum_{i\in\mathcal{I}^0} \tau_i v_i, \tau_i \in \mathbb{R}, i \in \mathcal{I}^0\} =$$

$$= \left\{ x : x = \tfrac{3}{13} \cdot \begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix} + \tau_1 \cdot \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}, \tau_1 \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{2}{13} + \frac{3}{2}\tau_1 \\ \frac{3}{13} + \tau_1 \end{pmatrix}, \tau_1 \in \mathbb{R} \right\},$$

calculate a representative minimum $x^* = \tau_2^* v_2 = \begin{pmatrix} -\frac{2}{13} \\ \frac{3}{13} \end{pmatrix}$,

calculate minimal value $f^* = -\dfrac{1}{4} \sum_{i\in\mathcal{I}^+} \dfrac{1}{\lambda_i\theta_i} (v_i^\top c)^2 = -1.$

- Geometrical interpretation



$$f(x) = 4x_1^2 - 12x_1x_2 + 9x_2^2 + 4x_1 - 6x_2,$$

set of multiple minima

$$X^* = \{(x_1, x_2) : x_1 = -\frac{2}{13} + \frac{3}{2}\tau_1, \\ x_2 = \frac{3}{13} + \tau_1, \\ \tau_1 \in (-\infty, +\infty)\}$$

representative minimum $x^* = \begin{pmatrix} -\frac{2}{13} \\ \frac{3}{13} \end{pmatrix}$,

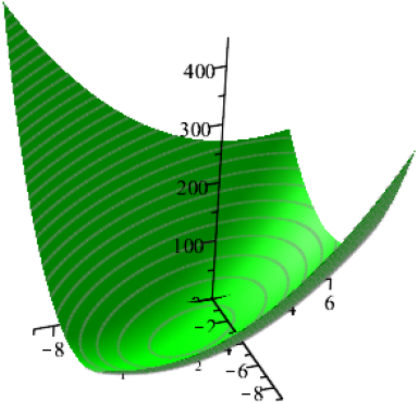
minimal value $f^* = -1$.

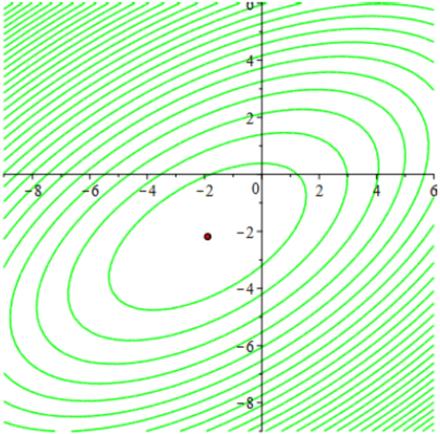
- Representing $f(x) = 4x_1^2 - 12x_1x_2 + 9x_2^2 + 4x_1 - 6x_2$ as a weighted sum of squares

$$f(x) = 9\left(-\frac{2}{3}x_1 + x_2 - \frac{1}{3}\right)^2 + 1.$$

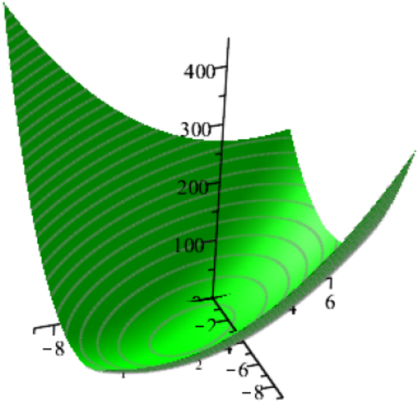
End of Example 4

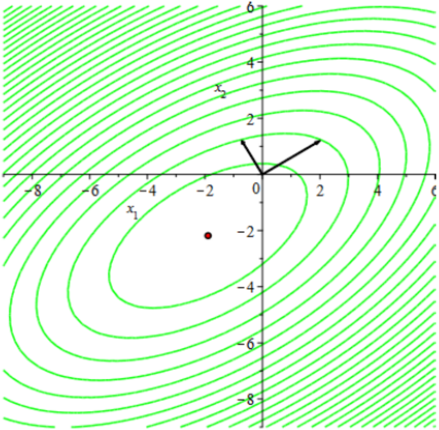
- Geometrical interpretation: $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$



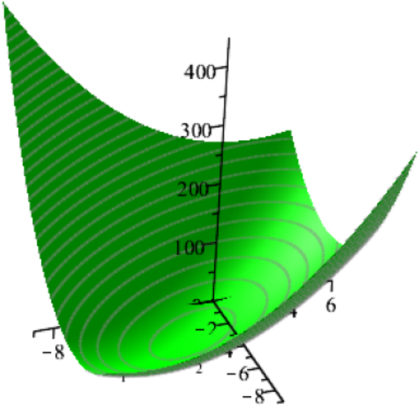


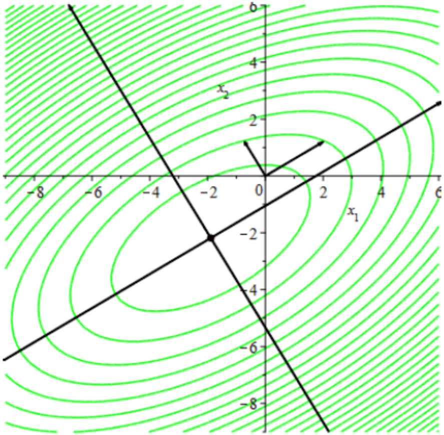
- Geometrical interpretation: $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$



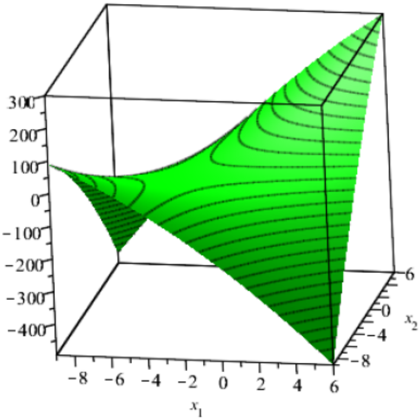


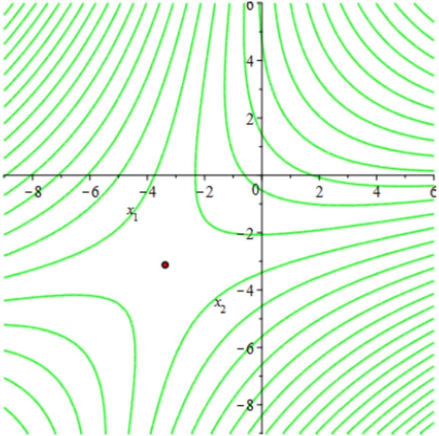
- Geometrical interpretation: $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$



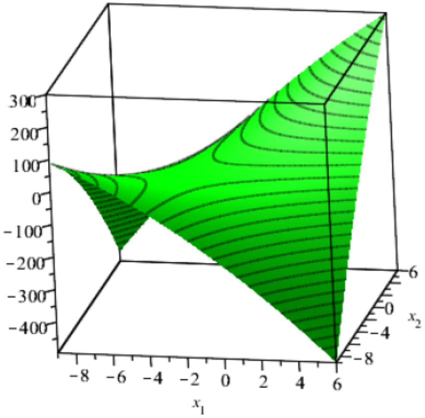


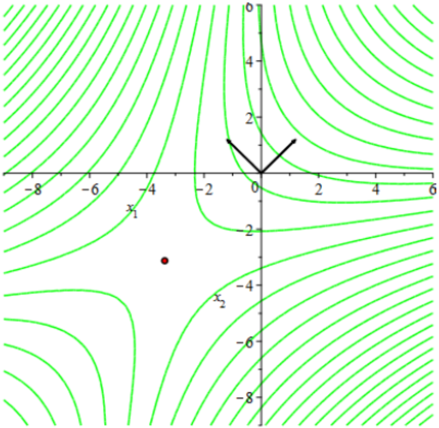
- Geometrical interpretation: $0 \leqslant \lambda_1 \leqslant \ldots \leqslant \lambda_k < 0 < \lambda_{k+1} \leqslant \ldots \leqslant \lambda_n$





- Geometrical interpretation: $0 \leqslant \lambda_1 \leqslant \ldots \leqslant \lambda_k < 0 < \lambda_{k+1} \leqslant \ldots \leqslant \lambda_n$





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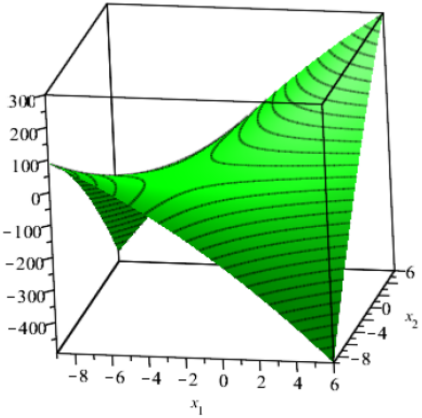

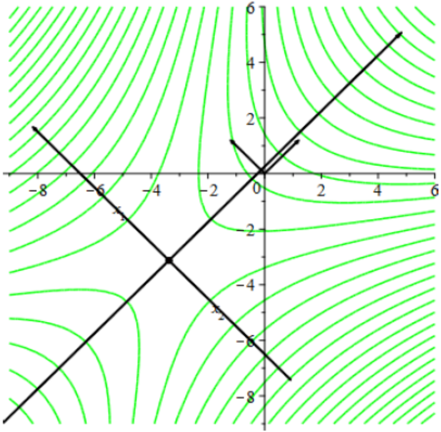

CHECK YOURSELF

Task 1. Find stationary point of quadratic function $f(x_1, x_2) = 6x_1^2 + 14x_2^2 + 6x_1x_2 - 60x_1 - 130x_2$. Determine the type of the stationary point.

Answer. Stationary point $x^* = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is a unique minimum with $f^* = f(x^*) = -350$.

Task 2. Find stationary point of quadratic function $f(x_1, x_2) = -11x_1^2 + 37x_2^2 + 20x_1x_2 - 58x_1 - 316x_2$. Determine the type of the stationary point.

Answer. Stationary point $x^* = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ is a unique saddle point with $f^* = f(x^*) = -661$.

Task 3. Find stationary point of quadratic function $f(x_1, x_2) = -22x_1^2 - 17x_2^2 - 12x_1x_2 - 76x_1 + 10x_2$. Determine the type of the stationary point.

Answer. Stationary point $x^* = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is a unique maximum with $f^* = f(x^*) = 81$.

Task 4. Find stationary point of quadratic function $f(x_1, x_2) = -4x_1^2 - 9x_2^2 - 12x_1x_2 + 4x_1 + 6x_2$. Determine the type of the stationary point.

Answer. The set of stationary points

$$X^* = \{(x_1, x_2) : 2x_1 + 3x_2 = 1\}.$$

All point in the line X^* are points of maximum with $f^* = f(x^*) = 1 \; \forall x^* \in X^*$.