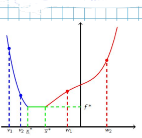
Properties of invex functions.

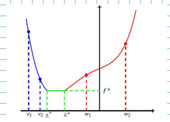
1. $f: R^n \to R$ is invex function
if $\nabla f(x_1)=0,\ \nabla f(x_2)=0 \Rightarrow$
$f(x_1)=f(x_2)=f^*,\ f^*=\min\{f(x): x\in R^n\}$



2. $f: R\to R$ is invex function. The set of global minima (the set of stationary points $X^*=\{x: f'(x)=0\}$ is convex, i.e. $X^*=[\underline{x}^*, \bar{x}^*]$.
(for univariate case only).

3. $f: R\to R$ is invex function and $X^*=[\underline{x}^*, \bar{x}^*]$ – the set of global minima. Then for any points $v_1<v_2<\underline{x}^*$ $f(v_1)>f(v_2)$
$\bar{x}^*<w_1<w_2$ $f(w_1)<f(w_2)$

Informal geometrical characteristic of univariate invex functions



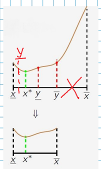
An univar. function f is invex iff there exists two points $\underline{x}^*, \bar{x}^* \in R$ such that $\underline{x}^*<\bar{x}^*$ and f is strictly monotone decreasing on $(-\infty, \underline{x}^*)$, it's constant on $[\underline{x}^*, \bar{x}^*]$, f is strictly monotone increasing on $[\bar{x}^*, +\infty)$. Each one ore two parts may be empty or degenerate into a single point.

Example: $f(x)=[\max\{0.5,\ x^4+x^3+0.3x+0.7\}]^2$

Decreasing part $(-\infty, \underline{x}^*)$
Constant part (global min) $[\underline{x}^*, \bar{x}^*]$
Increasing part $(\bar{x}^*, +\infty)$
$f^*=0.25$, $X^*=[-1.098, -0.477]$

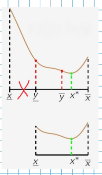
Interval reduction ($f(x^*)=\min f(x)$)

$f(x)$, $x\in[\underline{x}, \bar{x}]$, f – invex function,
$\underline{y}, \bar{y}\in[\underline{x}, \bar{x}]$, $\underline{y}<\bar{y}$

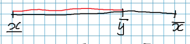


1. Right increasing $f(\underline{y})\le f(\bar{y}) \Rightarrow [\bar{y}, \bar{x}]\not\ni x^*$
$\Rightarrow$ reduction $[\bar{y}, \bar{x}]$, $\bar{x}=\bar{y}$

2. Left decreasing $f(\underline{y})\ge f(\bar{y}) \Rightarrow$
$[\underline{x}, \underline{y}]\not\ni x^*$, $\Rightarrow$ reduction $[\underline{x}, \underline{y}]$
$\underline{x}=\underline{y}$.



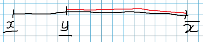
Choosing points $\underline{y}, \bar{y}$ Golden section algorithm.



$\frac{\bar{y}-\underline{x}}{\bar{x}-\underline{x}}=\delta$ $\quad\frac{\bar{x}-\bar{y}}{\bar{y}-\underline{x}}=\delta$

$\bar{y}=\underline{x}+\delta(\bar{x}-\underline{x})$

$\bar{x}-\underline{x}=\delta(\bar{x}-\underline{x})+\delta^2(\bar{x}-\underline{x})$
$(\bar{x}-\underline{x})=1$
$\delta^2+\delta-1=0$ $\quad\delta=\frac{\sqrt{5}-1}{2}\approx 0.618$
$\delta>0$

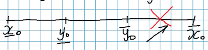


$\frac{\bar{x}-\underline{y}}{\bar{x}-\underline{x}}=\delta$ $\quad\frac{\underline{y}-\underline{x}}{\bar{x}-\underline{y}}=\delta$
$\bar{x}-\underline{y}=\delta(\bar{x}-\underline{x})$
$\underline{y}-\underline{x}=\delta(\bar{x}-\underline{y})$
$\underline{y}-\underline{x}=\delta^2(\bar{x}-\underline{x})=(1-\delta)(\bar{x}-\underline{x})$
$\underline{y}=\underline{x}+(1-\delta)(\bar{x}-\underline{x})$

Main properties of golden section

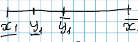
$\bar{y}_0=\underline{x}_0+\delta(\bar{x}_0-\underline{x}_0)$
$\underline{y}_0=\underline{x}_0+(1-\delta)(\bar{x}_0-\underline{x})$

1. Right increasing



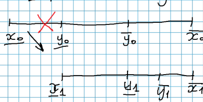
$f(\underline{y}_0)\le f(\bar{y}_0) \Rightarrow$ $\underline{x}_1=\underline{x}_0$, $\bar{x}_1=\bar{y}_0$

Check (proof):
$\bar{y}_1=\underline{x}_0+\delta(\bar{y}_0-\underline{x}_0)=\underline{x}+\delta^2(\bar{x}_0-\underline{x}_0)=$
$=\underline{x}_0+(1-\delta)(\bar{x}_0-\underline{x}_0)=\underline{y}_0$



$\bar{y}_1=\underline{x}_1+\delta(\bar{x}_1-\underline{x}_1)=\underline{y}_0$!
$\underline{y}_1=\underline{x}_1+(1-\delta)(\bar{x}_1-\underline{x}_1)$ – the only point to be computed

2. Left decreasing



$f(\underline{y}_0)\le f(\bar{y}_0) \Rightarrow$ $\underline{x}_1=\underline{y}_0$, $\bar{x}_1=\bar{x}_0$

$\underline{y}_1=\underline{x}_1+(1-\delta)(\bar{x}_1-\underline{x}_1)=\bar{y}_0$!
$\bar{y}_1=\underline{x}_1+\delta(\bar{x}_1-\underline{x}_1)$ –
the only point to be computed.

Golden section algorithm

```
findMin(f, start, end, ε) {          ← solution accuracy
  x̲ = start; x̄ = end; k = 0;          ← number of iterations      } initialization
  δ = (√5−1)/2; y̲ = x̲ + (1−δ)(x̄−x̲); ȳ = x̲ + δ(x̄−x̲);
  f̲ = f(y̲); f̄ = f(ȳ);
  while (x̄ − x̲ > ε) {                 ε = 0.0001
    if (f̲ > f̄) { // left decreasing
      x̲ = y̲; y̲ = ȳ; f̲ = f̄;
      ȳ = x̲ + δ(x̄ − x̲); f̄ = f(ȳ);
    }
    else { // right increasing
      x̄ = ȳ; ȳ = y̲; f̄ = f̲;
      y̲ = x̲ + (1−δ)(x̄ − x̲); f̲ = f(y̲);
    }
    k = k+1;
  } // end of while
  x* = x̲; f* = f(x̲);
  return {x*, f*, k};
} // stop
```