- Example 1. For given function $f(x_1, x_2) = e^{x_1^2 - x_2}(5 - 2x_1 + x_2)$ find all stationary points, points of local minimum, points of local maximum, minimal and maximal values.

- *Solution.* First order necessary optimality condition $\nabla f(x) = 0$:

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f(x_1,x_2)}{\partial x_1} \\ \frac{f(x_1,x_2)}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2x_1 e^{x_1^2-x_2}(5-2x_1+x_2) - 2e^{x_1^2-x_2} \\ -e^{x_1^2-x_2}(5-2x_1+x_2) + e^{x_1^2-x_2} \end{pmatrix} = e^{x_1^2-x_2}\begin{pmatrix} 2x_1(5-2x_1+x_2)-2 \\ -(5-2x_1+x_2)+1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $e^{x_1^2-x_2} > 0 \ \forall x_1, \ \forall x_2$

$$\nabla f(x) = 0 \ \Leftrightarrow \ \begin{pmatrix} 2x_1(5-2x_1+x_2)-2 \\ -(5-2x_1+x_2)+1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ \Leftrightarrow \begin{cases} -4x_1^2 + (2x_2+10)x_1 - 2 = 0 & (1) \\ 2x_1 - x_2 - 4 = 0 & (2) \end{cases}$$

From (2) $\Rightarrow x_2 = 2x_1 - 4$ substituting in (1) $\Rightarrow 2x_1 - 2 = 0 \Rightarrow x_1 = 1 \Rightarrow x_2 = -2.$

Unique stationary point $x^* = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

- Example 1. For given function $f(x_1, x_2) = e^{x_1^2 - x_2}(5 - 2x_1 + x_2)$ find all stationary points, points of local minimum, points of local maximum, minimal and maximal values.

- *Solution.* Second order optimality condition at $x^* = (1, -2)^\top$.

$$\nabla^2 f(x) = \begin{pmatrix} 2\,e^{x_1^2 - x_2}\,(5 - 2\,x_1 + x_2) + 4\,x_1^2 e^{x_1^2 - x_2}\,(5 - 2\,x_1 + x_2) - 8\,x_1\,e^{x_1^2 - x_2} & -2\,x_1\,e^{x_1^2 - x_2}\,(5 - 2\,x_1 + x_2) + 2\,x_1\,e^{x_1^2 - x_2} + 2\,e^{x_1^2 - x_2} \\ -2\,x_1\,e^{x_1^2 - x_2}\,(5 - 2\,x_1 + x_2) + 2\,x_1\,e^{x_1^2 - x_2} + 2\,e^{x_1^2 - x_2} & e^{x_1^2 - x_2}\,(5 - 2\,x_1 + x_2) - 2\,e^{x_1^2 - x_2} \end{pmatrix}$$

$$\nabla^2 f(x^*) = \begin{pmatrix} -2e^3 & 2e^3 \\ 2e^3 & -e^3 \end{pmatrix} = \begin{pmatrix} -40.17107384 & 40.17107384 \\ 40.17107384 & -20.08553692 \end{pmatrix}$$

Eigenvalues $\lambda_1 = -71.5357005142016 < 0$, $\lambda_2 = 11.2790897542016 > 0 \Rightarrow$

$\Rightarrow \nabla^2 f(x^*)$ is indefinite $\Rightarrow x^* = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is a saddle point,

no points of local minimum or maximum.

- Example 1. For given function $f(x_1, x_2) = e^{x_1^2 - x_2}(5 - 2x_1 + x_2)$ find all stationary points, points of local minimum, points of local maximum, minimal and maximal values.

- *Solution.* Analysing inf and sup behaviour.

Line 1: $L_1 = \{(x_1, x_2) : 5 - 2x_1 + x_2 = 0\} \Rightarrow f(x_1, x_2) = e^{x_1^2 - x_2} \cdot 0 = 0,\ (x_1, x_2) \in L_1.$

Line 2: $L_2 = \{(x_1, x_2) : 5 - 2x_1 + x_2 = 1\} \Rightarrow f(x_1, x_2) = e^{x_1^2 - x_2} \cdot 1 = e^{x_1^2 - x_2},\ (x_1, x_2) \in L_2$

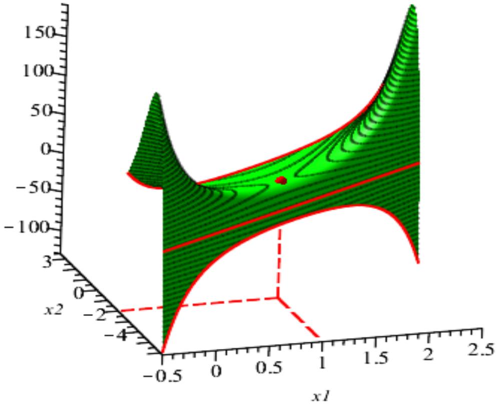
$$f(x_1, x_2 = 2x_1 - 4) = e^{x_1^2 - 2x_1 + 4} = e^{(x_1 - 1)^2 + 3} \to +\infty \text{ as } x_1 \to +\infty \text{ (unbounded from above)}$$

Line 3: $L_3 = \{(x_1, x_2) : 5 - 2x_1 + x_2 = -1\} \Rightarrow f(x_1, x_2) = e^{x_1^2 - x_2} \cdot (-1) = -e^{x_1^2 - x_2},\ (x_1, x_2) \in L_3$

$$f(x_1, x_2 = 2x_1 - 6) = -e^{x_1^2 - 2x_1 + 6} = -e^{(x_1 - 1)^2 + 5} \to -\infty \text{ as } x_1 \to +\infty \text{ (unbounded from below)}$$

- *Answer.* f has unique saddle point $x^* = (1, -2)^\top$, is unbounded from above $\sup\limits_{x \in \mathbb{R}^2} f(x) = +\infty$, and is unbounded from below $\inf\limits_{x \in \mathbb{R}^2} f(x) = -\infty$.

- Example 1. For given function $f(x_1, x_2) = e^{x_1^2 - x_2}(5 - 2x_1 + x_2)$ find all stationary points, points of local minimum, points of local maximum, minimal and maximal values.
- Geometrical interpretation.



- Example 2. For given function $f(x_1, x_2) = \mathrm{e}^{2\,x_1+3\,x_2}\left(8\,x_1{}^2 - 6\,x_1\,x_2 + 3\,x_2{}^2\right)$ find all stationary points, points of local minimum, points of local maximum, minimal and maximal values.

- *Solution.* First order necessary optimality condition $\nabla f(x) = 0$:

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f(x_1,x_2)}{\partial x_1} \\ \frac{f(x_1,x_2)}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2\,\mathrm{e}^{2\,x_1+3\,x_2}\left(8\,x_1{}^2 - 6\,x_1\,x_2 + 3\,x_2{}^2\right) + \mathrm{e}^{2\,x_1+3\,x_2}\left(16\,x_1 - 6\,x_2\right) \\ 3\,\mathrm{e}^{2\,x_1+3\,x_2}\left(8\,x_1{}^2 - 6\,x_1\,x_2 + 3\,x_2{}^2\right) + \mathrm{e}^{2\,x_1+3\,x_2}\left(-6\,x_1 + 6\,x_2\right) \end{pmatrix} =$$

$$= e^{2\,x_1+3\,x_2}\begin{pmatrix} 16\,x_1{}^2 - 12\,x_1\,x_2 + 6\,x_2{}^2 + 16\,x_1 - 6\,x_2 \\ 24\,x_1{}^2 - 18\,x_1\,x_2 + 9\,x_2{}^2 - 6\,x_1 + 6\,x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $e^{2x_1+3x_2} > 0\ \forall x_1,\ \forall x_2$

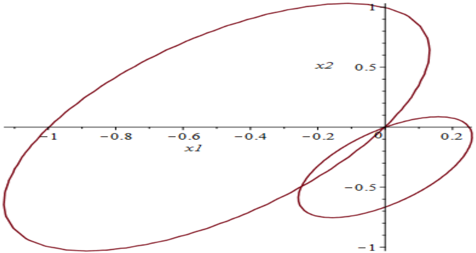
$$\begin{cases} 16\,x_1{}^2 - 12\,x_1\,x_2 + 6\,x_2{}^2 + 16\,x_1 - 6\,x_2 = 0, \\ 24\,x_1{}^2 - 18\,x_1\,x_2 + 9\,x_2{}^2 - 6\,x_1 + 6\,x_2 = 0. \end{cases}$$

- Example 2. For given function $f(x_1, x_2) = e^{2x_1+3x_2}\left(8x_1^2 - 6x_1x_2 + 3x_2^2\right)$ find all stationary points, points of local minimum, points of local maximum, minimal and maximal values.

- *Solution.* First order necessary optimality condition $\nabla f(x) = 0$:

$$\begin{cases} q_1(x_1, x_2) = 16x_1^2 - 12x_1x_2 + 6x_2^2 + 16x_1 - 6x_2 = 0, \\ q_2(x_1, x_2) = 24x_1^2 - 18x_1x_2 + 9x_2^2 - 6x_1 + 6x_2 = 0. \end{cases}$$

Geometrical interpretation.



- Example 2. For given function $f(x_1, x_2) = \mathrm{e}^{2\,x_1+3\,x_2}\left(8\,x_1{}^2 - 6\,x_1\,x_2 + 3\,x_2{}^2\right)$ find all stationary points, points of local minimum, points of local maximum, minimal and maximal values.
- *Solution.* First order necessary optimality condition $\nabla f(x) = 0$:

$$\begin{cases} q_1(x_1, x_2) = & 16\,x_1{}^2 - 12\,x_1\,x_2 + 6\,x_2{}^2 + 16\,x_1 - 6\,x_2 = 0, \\ q_2(x_1, x_2) = & 24\,x_1{}^2 - 18\,x_1\,x_2 + 9\,x_2{}^2 - 6\,x_1 + 6\,x_2 = 0. \quad \cdot\left(-\frac{2}{3}\right) \end{cases}$$

$$\left(-16\,x_1{}^2 + 12\,x_1\,x_2 - 6\,x_2{}^2 + 4\,x_1 - 4\,x_2\right)$$

$$\begin{aligned} 20x_1 - 10x_2 &= 0, \\ 2x_1 - \quad x_2 &= 0. \end{aligned} \qquad (3)$$

From (3) $x_1 = \frac{1}{2}x_2 \Rightarrow q_2(x_1 = \frac{1}{2}x_2, x_2) = 6x_2^2 + 3x_2 = 3x_2(2x_2 + 1) = 0 \Rightarrow$

$\Rightarrow x_2^{*,1} = 0,\ x_2^{*,2} = -\frac{1}{2} \Rightarrow x_1^{*,1} = 0,\ x_1^{*,2} = -\frac{1}{4}.$

Two stationary points $x^{*,1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad x^{*,2} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{4} \end{pmatrix}.$

- Example 2. For given function $f(x_1, x_2) = \mathrm{e}^{2\,x_1+3\,x_2}\left(8\,x_1{}^2 - 6\,x_1\,x_2 + 3\,x_2{}^2\right)$ find all stationary points, points of local minimum, points of local maximum, minimal and maximal values.
- *Solution.* Second order optimality condition

$$\frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_1} = 4\,\mathrm{e}^{2\,x1+3\,x2}\left(8\,x1^2 - 6\,x1\,x2 + 3\,x2^2\right) + 4\,\mathrm{e}^{2\,x1+3\,x2}\,(16\,x1 - 6\,x2) + 16\,\mathrm{e}^{2\,x1+3\,x2}$$

$$\frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2} = \frac{\partial^2 f(x_1,x_2)}{\partial x_2 \partial x_1} = 6\,\mathrm{e}^{2\,x1+3\,x2}\left(8\,x1^2 - 6\,x1\,x2 + 3\,x2^2\right) + 2\,\mathrm{e}^{2\,x1+3\,x2}\,(-6\,x1 + 6\,x2) + 3\,\mathrm{e}^{2\,x1+3\,x2}\,(16\,x1 - 6\,x2) - 6\,\mathrm{e}^{2\,x1+3\,x2}$$

$$\frac{\partial^2 f(x_1,x_2)}{\partial x_2 \partial x_2} = 9\,\mathrm{e}^{2\,x1+3\,x2}\left(8\,x1^2 - 6\,x1\,x2 + 3\,x2^2\right) + 6\,\mathrm{e}^{2\,x1+3\,x2}\,(-6\,x1 + 6\,x2) + 6\,\mathrm{e}^{2\,x1+3\,x2}$$

$$\nabla^2 f(x^{*,1}) = \begin{pmatrix} 16 & -6 \\ -6 & 6 \end{pmatrix} \Rightarrow \lambda_1 = 18.81024968 > 0,\quad \lambda_2 = 3.189750324 > 0 : \underline{\text{isolated loc.min.}}$$

$$\nabla^2 f(x^{*,2}) = \begin{pmatrix} -0.7385392599 & -2.411113466 \\ -2.411113466 & 4.854809547 \end{pmatrix} \Rightarrow \lambda_1 = -1.634405427 < 0,\quad \lambda_2 = 5.750675714 > 0 :$$

$\underline{\text{saddle point}}$