- Unconstrained minimization

$$\min f(x),$$

$$x \in \mathbb{R}^n.$$

- First order necessary optimality condition:

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{pmatrix} = 0.$$

- Point x^*: $\nabla f(x^*) = 0$ is called a *stationary* point.

UNCONSTRAINED QUADRATIC OPTIMIZATION

- Unconstrained quadratic optimization: objective function is quadratic, no constraints

$$\min_{x\in\mathbb{R}^n}\{f(x) = x^\top Qx + c^\top x\},$$

$$Q = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix} \in \mathbb{R}^{n\times n}, \quad c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n,$$

Q assumed to be symmetric: $q_{ij} = q_{ji}$, otherwise you may set

$$Q \leftarrow \tfrac{1}{2}(Q + Q^\top).$$

(The latter means the following: $x^\top Qx = \frac{1}{2}x^\top(Q + Q^\top)x$ and $Q + Q^\top$ is always symmetric).

- Unconstrained quadratic optimization: objective function is quadratic, no constraints

$$\min_{x\in\mathbb{R}^n}\{f(x) = x^\top Qx + c^\top x\},$$

- Extracting Q and c:

Example 1. $f(x_1, x_2) = x_1^2 - x_2^2 - 4x_1 + 6x_2 \;\Rightarrow\; Q = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\ c = \begin{pmatrix}-4\\ 6\end{pmatrix};$

$$f(x) = \underbrace{\begin{pmatrix}x_1 & x_2\end{pmatrix}}_{x^\top}\underbrace{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}_{Q}\underbrace{\begin{pmatrix}x_1\\ x_2\end{pmatrix}}_{x} + \underbrace{\begin{pmatrix}-4 & 6\end{pmatrix}}_{c^\top}\underbrace{\begin{pmatrix}x_1\\ x_2\end{pmatrix}}_{x}$$

- Unconstrained quadratic optimization: objective function is quadratic,
  no constraints

$$\min_{x \in \mathbb{R}^n} \{f(x) = x^\top Q x + c^\top x\},$$

- Extracting Q and c:

Example 2. $f(x_1, x_2) = 5x_1^2 + 4x_1x_2 + x_2^2 - 16x_1 - 12x_2 \Rightarrow Q = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}, c = \begin{pmatrix} -16 \\ -12 \end{pmatrix};$

$$f(x) = \underbrace{\begin{pmatrix} x_1 & x_2 \end{pmatrix}}_{x^\top} \underbrace{\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} -16 & -12 \end{pmatrix}}_{c^\top} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x}$$

- Unconstrained quadratic optimization: objective function is quadratic, no constraints

$$\min_{x \in \mathbb{R}^n} \{f(x) = x^\top Q x + c^\top x\},$$

- Extracting Q and c:

Example 3. $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + 2x_3^2 + x_1 x_2 + 2x_1 x_3 + 3x_2 x_3 - x_1$

$$f(x) = \underbrace{\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}}_{x^\top} \underbrace{\begin{pmatrix} 1 & \frac{1}{2} & 1 \\ \frac{1}{2} & 1 & \frac{3}{2} \\ 1 & \frac{3}{2} & 2 \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} -1 & 0 & 0 \end{pmatrix}}_{c^\top} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x}$$

- Unconstrained quadratic optimization: objective function is quadratic, no constraints

$$\min_{x\in\mathbb{R}^n}\{f(x) = x^\top Qx + c^\top x\},$$

- Eigenstructure: $Q \sim (\Lambda, V)$, $Qv_i = \lambda_i v_i$, $i = 1, \dots, n$.

! Q is symmetric matrix $\Rightarrow$ eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \dots \leqslant \lambda_n$ are real numbers

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}, \quad V = \begin{pmatrix} \boxed{\begin{matrix} v_{11} \\ v_{12} \\ \vdots \\ v_{1n} \end{matrix}} & \boxed{\begin{matrix} v_{21} \\ v_{22} \\ \vdots \\ v_{2n} \end{matrix}} & \begin{matrix} \cdots \\ \cdots \\ \vdots \\ \cdots \end{matrix} & \boxed{\begin{matrix} v_{n1} \\ v_{n2} \\ \vdots \\ v_{nn} \end{matrix}} \end{pmatrix},$$

$$\begin{matrix} \uparrow & \uparrow & & \uparrow \\ v_1 & v_2 & & v_n \end{matrix}$$

Λ - diagonal matrix with eigenvalues on the main diagonal,

V - matrix with eigenvectors $v_1, v_2, \dots, v_n$ as columns.

- Matrix classification
- Q is said to be Positive Definite (PD, $Q \succ 0$) if $x^\top Qx > 0 \; \forall x \in \mathbb{R}^n$, $x \neq 0$.
  (If Q is symmetric, $Q = Q^\top$, then PD $\sim \lambda_i > 0, i = 1, \ldots, n$.)
- Q is said to be Positive SemiDefinite (PSD, $Q \succeq 0$) if $x^\top Qx \geqslant 0 \; \forall x \in \mathbb{R}^n$.
  (If Q is symmetric, $Q = Q^\top$, then PSD $\sim \lambda_i \geqslant 0, i = 1, \ldots, n$.)
- Q is said to be Negative SemiDefinite (NSD, $Q \preceq 0$) if $x^\top Qx \leqslant 0 \; \forall x \in \mathbb{R}^n$.
  (If Q is symmetric, $Q = Q^\top$, then NSD $\sim \lambda_i \leqslant 0, i = 1, \ldots, n$.)
- Q is said to be Negative Definite (ND, $Q \prec 0$) if $x^\top Qx < 0 \; \forall x \in \mathbb{R}^n$, $x \neq 0$.
  (If Q is symmetric, $Q = Q^\top$, then ND $\sim \lambda_i < 0, i = 1, \ldots, n$.)
- Otherwise, Q is said to be indefinite.

- Quadratic form: $f(x) = x^\top Qx$

- Stationary point: $\nabla f(x) = 2Qx = 0 \;\Rightarrow\; x^* = 0$ - trivial!

Example 4. $f(x_1, x_2) = x_1^2 + x_2^2$

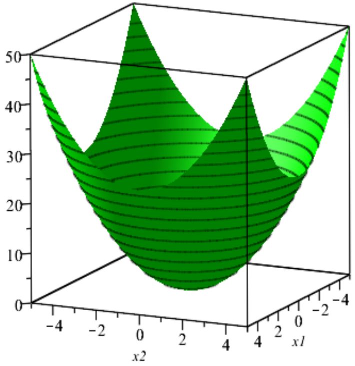
$$\text{or } f(x_1, x_2) = 0.5x_1^2 + 2x_2^2$$

$$\text{or } f(x_1, x_2) = 5x_1^2 + 0.1x_2^2$$

$$\Downarrow$$

$$f(x_1, x_2) = \lambda_1 x_1^2 + \lambda_2 x_2^2$$

$\lambda_1 > 0,\ \lambda_2 > 0 \;\Rightarrow\; x^* = 0$ - unique minimum



- Quadratic form: $f(x) = x^\top Qx$

- Stationary point: $\nabla f(x) = 2Qx = 0 \;\Rightarrow\; x^* = 0$ - trivial!

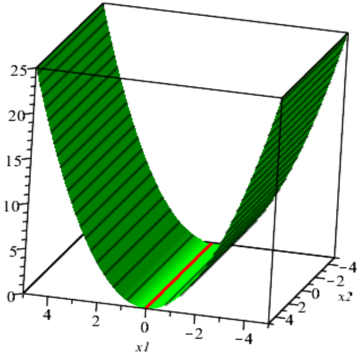
Example 5. $f(x_1, x_2) = x_1^2$

or $f(x_1, x_2) = 0.5x_1^2$

or $f(x_1, x_2) = 0.1x_2^2$

$$\Downarrow$$

$$f(x_1, x_2) = \lambda_1 x_1^2 + \lambda_2 x_2^2$$



$\lambda_1 > 0, \lambda_2 = 0$
$\lambda_1 = 0, \lambda_2 > 0$ $\Rightarrow x^* = 0$ - nonunique minimum

- Quadratic form: $f(x) = x^\top Qx$

- Stationary point: $\nabla f(x) = 2Qx = 0 \;\Rightarrow\; x^* = 0$ - trivial!

Example 6. $f(x_1, x_2) = x_1^2 - x_2^2$

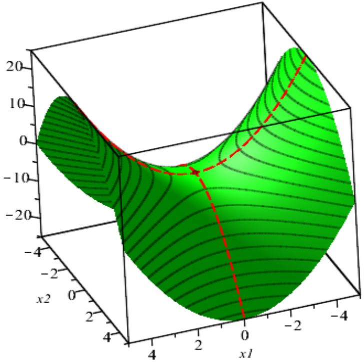
or $f(x_1, x_2) = -x_1^2 + 0.5x_1^2$

or $f(x_1, x_2) = 0.1x_1^2 - 8x_2^2$

$$\Downarrow$$

$$f(x_1, x_2) = \lambda_1 x_1^2 + \lambda_2 x_2^2$$

$$\left.\begin{array}{l}\lambda_1 > 0, \lambda_2 < 0 \\ \lambda_1 < 0, \lambda_2 > 0\end{array}\right. \Rightarrow x^* = 0 \text{ - saddle point}$$



- Quadratic form: $f(x) = x^\top Qx$

- Stationary point: $\nabla f(x) = 2Qx = 0 \;\Rightarrow\; x^* = 0$ - trivial!

Example 7. $f(x_1, x_2) = -x_1^2$

or $f(x_1, x_2) = -0.5x_2^2$

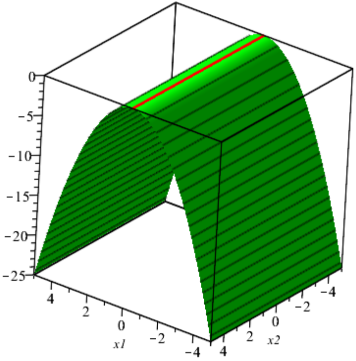
or $f(x_1, x_2) = -100x_1^2$

$$\Downarrow$$

$$f(x_1, x_2) = \lambda_1 x_1^2 + \lambda_2 x_2^2$$

$\lambda_1 < 0, \lambda_2 = 0$

$\lambda_1 = 0, \lambda_2 < 0$ $\Rightarrow x^* = 0$ - nonunique maximun



- Quadratic form: $f(x) = x^\top Q x$

- Stationary point: $\nabla f(x) = 2Qx = 0 \;\Rightarrow\; x^* = 0$ - trivial!

Example 8. $f(x_1, x_2) = -x_1^2 - x_2^2$

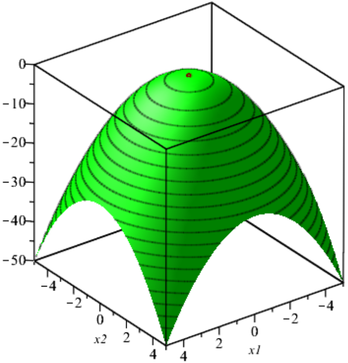
$$\text{or } f(x_1, x_2) = -5x_1^2 - 0.5x_2^2$$

$$\text{or } f(x_1, x_2) = -0.1x_1^2 - 10x_2^2$$

$$\Downarrow$$

$$f(x_1, x_2) = \lambda_1 x_1^2 + \lambda_2 x_2^2$$

$\lambda_1 < 0, \lambda_2 < 0 \;\Rightarrow x^* = 0$ - unique maximum



- Quadratic form: $f(x) = x^\top Qx$

- Stationary point: $\nabla f(x) = 2Qx = 0 \;\Rightarrow\; x^* = 0$ - trivial!

- Separable form $=$ Q is diagonal

- $f(x) = \sum_{j=1}^{n} \lambda_j x_j^2$

$$Q = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

- .1 $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$ $x^* = 0$ - unique minimum;
- .2 $0 = \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$ $x^* = 0$ - nonunique minimum ($\lambda_n > 0$);
- .3 $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_k < 0 < \lambda_{k+1} \leqslant \ldots \leqslant \lambda_n$ $x^* = 0$ - saddle point;
- .4 $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \lambda_n = 0$ $x^* = 0$ - nonunique maximum ($\lambda_1 < 0$);
- .5 $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \lambda_n < 0$ $x^* = 0$ - unique maximum.

- Quadratic form: $f(x) = x^\top Q x$

- Stationary point: $\nabla f(x) = 2Qx = 0 \;\Rightarrow\; x^* = 0$ - trivial!

- General case

$$Q = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}$$

Example 9. $f(x_1, x_2) = x_1^2 - x_1 x_2 + x_2^2$

Nevertheless! $f(x_1, x_2) = \frac{1}{4}(x_1 + x_2)^2 + \frac{3}{4}(-x_1 + x_2)^2 \;\geqslant\; 0$

$$\Downarrow$$

$$x^* = 0 \text{ is a minimum point, } \nabla f(x) = \begin{pmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 \end{pmatrix} = 0.$$

- Quadratic form: $f(x) = x^\top Qx$
- Stationary point: $\nabla f(x) = 2Qx = 0 \;\Rightarrow\; x^* = 0$ - trivial!
- General case

$$Q = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}$$

Example 10. $f(x_1, x_2) = -x_1x_2 + x_2^2$

Nevertheless! $f(x_1, x_2) = -0.207(0.924x_1 - 0.383x_2)^2 + 1.207(0.383x_1 + 0.924x_2)^2$

$$\Downarrow$$

$$x^* = 0 \text{ is a saddle point, } \nabla f(x) = \begin{pmatrix} -x_2 \\ -x_1 + 2x_2 \end{pmatrix} = 0.$$

- Quadratic form: $f(x) = x^\top Q x$
- Stationary point: $\nabla f(x) = 2Qx = 0 \;\Rightarrow\; x^* = 0$ - trivial!
- General case

$$Q = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}$$

Example 11. $f(x_1, x_2) = -x_1^2 - x_1 x_2 - x_2^2$

Nevertheless! $f(x_1, x_2) = -0.25(x_1 + x_2)^2 - 0.75(-x_1 + x_2)^2$

$$\Downarrow$$

$$x^* = 0 \text{ is a maximum point, } \nabla f(x) = \begin{pmatrix} -2x_1 - x_2 \\ -x_1 - 2x_2 \end{pmatrix} = 0.$$

- Quadratic form: $f(x) = x^\top Qx$
- Stationary point: $\nabla f(x) = 2Qx = 0 \;\Rightarrow\; x^* = 0$ - trivial!
- Eigenstructure analysis: *if* $\hat{\lambda}$ *and* $\hat{v}$ are such that

$$Q\hat{v} = \hat{\lambda}\hat{v}$$

then $\hat{\lambda}$ *is called an eigenvalue of matrix* Q *and*

$\hat{v}$ *is called an eigenvector of matrix* Q.

- In order to find an eigenpair $(\hat{\lambda}, \hat{v})$ one has to solve

$$Qx = \lambda x \;\Rightarrow\; (Q - \lambda I)x = 0.$$

- Nontrivial solutions iff (if and only if)

$$\det(Q - \lambda I) = 0.$$

- $\det(Q - \lambda I)$ is a polynomial of the degree n.

- If Q is symmetric then $\det(Q - \lambda I)$ has n real roots (eigenvalues):

$$\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n.$$

- Unconstrained quadratic optimization: objective function is quadratic, no constraints

$$\min_{x \in \mathbb{R}^n} \{ f(x) = x^\top Q x + c^\top x \},$$

- Eigenstructure: $Q \sim (\Lambda, V)$, $Qv_i = \lambda_i v_i$, $i = 1, \ldots, n$.

! Q is symmetric matrix $\Rightarrow$ eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$ are real numbers

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}, \quad V = \begin{array}{c} \begin{pmatrix} \boxed{\begin{matrix} v_{11} \\ v_{12} \\ \vdots \\ v_{1n} \end{matrix}} & \boxed{\begin{matrix} v_{21} \\ v_{22} \\ \vdots \\ v_{2n} \end{matrix}} & \begin{matrix} \cdots \\ \cdots \\ \vdots \\ \cdots \end{matrix} & \boxed{\begin{matrix} v_{n1} \\ v_{n2} \\ \vdots \\ v_{nn} \end{matrix}} \end{pmatrix} \\ \begin{matrix} \uparrow & \quad \uparrow & \qquad\quad & \uparrow \\ v_1 & \quad v_2 & \qquad\quad & v_n \end{matrix} \end{array},$$

Λ - diagonal matrix with eigenvalues on the main diagonal,
V - matrix with eigenvectors $v_1, v_2, \ldots, v_n$ as columns.

- Unconstrained quadratic optimization: objective function is quadratic, no constraints

$$\min_{x\in\mathbb{R}^n}\{f(x) = x^\top Qx + c^\top x\},$$

- Eigenstructure: $Q \sim (\Lambda, V)$, $Qv_i = \lambda_i v_i$, $i = 1, \ldots, n$,

$$\boxed{QV = V\Lambda \qquad (1)}$$

Eigenvectors $v_1, v_2, \ldots, v_n$ are orthogonal:

$$v_i^\top v_j = \begin{cases} \theta_i, & i = j, \\ 0, & i \neq j \end{cases}, \quad \theta_j = \|v_j\|^2 > 0, \quad \Theta = \begin{pmatrix} \theta_1 & 0 & \cdots & 0 \\ 0 & \theta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \theta_n \end{pmatrix}$$

$$\boxed{V^\top V = \Theta \qquad (2)}$$

$\boxed{\text{From (1), (2)}} \Rightarrow$ $\boxed{V^\top QV = V^\top V\Lambda = \Theta\Lambda \qquad (3)}$

- Unconstrained quadratic optimization: objective function is quadratic, no constraints

$$\min_{x\in\mathbb{R}^n}\{f(x)=x^\top Qx+c^\top x\},$$

- Inveres matrices:

$$\boxed{V^\top V=\Theta \quad (2)} \Rightarrow \quad V^\top V=\Theta^{\frac{1}{2}}\Theta^{\frac{1}{2}} \;\Rightarrow \boxed{\Theta^{-\frac{1}{2}}V^\top V\Theta^{-\frac{1}{2}}=\mathrm{I}, \qquad (4)}$$

$$\Theta^{\frac{1}{2}}=\begin{pmatrix}\sqrt{\theta_1} & 0 & \cdots & 0\\ 0 & \sqrt{\theta_2} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \sqrt{\theta_n}\end{pmatrix},\Theta^{-\frac{1}{2}}=\begin{pmatrix}\frac{1}{\sqrt{\theta_1}} & 0 & \cdots & 0\\ 0 & \frac{1}{\sqrt{\theta_2}} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \frac{1}{\sqrt{\theta_n}}\end{pmatrix},\mathrm{I}=\begin{pmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{pmatrix}$$

$$\boxed{\begin{array}{r}\text{From } (4) \text{ and}\\ (AB)^{-1}=B^{-1}A^{-1}\end{array}} \Rightarrow \begin{cases}\left(\Theta^{-\frac{1}{2}}V^\top\right)^{-1}=\left(V^\top\right)^{-1}\Theta^{\frac{1}{2}}=V\Theta^{-\frac{1}{2}} \Rightarrow \boxed{\left(V^\top\right)^{-1}=V\Theta^{-1}, \quad (5)}\\ \left(V\Theta^{-\frac{1}{2}}\right)^{-1} = \Theta^{\frac{1}{2}}V^{-1}=\Theta^{-\frac{1}{2}}V^\top \Rightarrow \boxed{V^{-1}=\Theta^{-1}V^\top, \quad (6)}\end{cases}$$

- Unconstrained quadratic optimization: objective function is quadratic, no constraints

$$\min_{x\in\mathbb{R}^n}\{f(x) = x^\top Qx + c^\top x\},$$

- Decomposition of Q:

$$\boxed{\text{From } V^\top QV = \Theta\Lambda \text{ (see (3)), (5), (6)}} \Rightarrow\ Q = \underbrace{V\Theta^{-1}}_{(V^\top)^{-1}}\Theta\Lambda\underbrace{\Theta^{-1}V^\top}_{V^{-1}} = V\Lambda\Theta^{-1}V^\top$$

$$Q = V\Lambda\Theta^{-1}V^\top$$

- Then $$\boxed{f(x) = x^\top V\Lambda\Theta^{-1}V^\top x + c^\top x \qquad (7)}$$
- Variable substitutions: $y = V^\top x \ \Leftrightarrow\ x = V\Theta^{-1}y$ (see (5))
- New objective function: $\tilde{f}(y) = y^\top\Lambda\Theta^{-1}y + d^\top y,\ \ d = \Theta^{-1}V^\top c.$

- Unconstrained quadratic optimization: reduction to the separable problem

$$\boxed{\min_{x\in\mathbb{R}^n}\{f(x) = x^\top Q x + c^\top x\}}$$

$$\Downarrow \qquad\qquad\qquad \Uparrow$$

$$\boxed{x = V\Theta^{-1}y} \qquad \boxed{y = V^\top x}$$

$$\Downarrow \qquad\qquad\qquad \Uparrow$$

$$\boxed{\min_{y\in\mathbb{R}^n}\{\tilde{f}(y) = y^\top \Lambda\Theta^{-1}y + d^\top y\}}$$

- Advantage: $\tilde{f}$ is a separable function

$$\tilde{f}(y) = \sum_{i=1}^{n}\left(\frac{\lambda_i}{\theta_i}y_i^2 + d_i y_i\right), \qquad d_i = \frac{1}{\theta_i}v_i^\top c \tag{8}$$

- Separable objective:

$$\tilde{f}(y) = \sum_{i=1}^{n} \left(\frac{\lambda_i}{\theta_i} y_i^2 + d_i y_i \right)$$

- Index sets: $\mathcal{I}^- = \{i : \lambda_i < 0\}$, $\mathcal{I}^0 = \{i : \lambda_i = 0\}$, $\mathcal{I}^+ = \{i : \lambda_i > 0\}$
- Then

$$\tilde{f}(y) = \sum_{i \in \mathcal{I}^-} \left(\frac{\lambda_i}{\theta_i} y_i^2 + d_i y_i \right) + \sum_{i \in \mathcal{I}^0} d_i y_i + \sum_{i \in \mathcal{I}^+} \left(\frac{\lambda_i}{\theta_i} y_i^2 + d_i y_i \right)$$

$$\Downarrow$$

$$\boxed{\tilde{f}(y) = \sum_{i \in \mathcal{I}^- \cup \mathcal{I}^+} \frac{\lambda_i}{\theta_i} \left(y_i + \frac{d_i \theta_i}{2\lambda_i} \right)^2 + \sum_{i \in \mathcal{I}^0} d_i y_i - \frac{1}{4} \sum_{i \in \mathcal{I}^- \cup \mathcal{I}^+} \frac{d_i^2 \theta_i}{\lambda_i} \qquad (9)}$$